NUMBER & ALGEBRA

CHAPTER 17

Complex Numbers

17

SYLLABUS 1.12–1.14 HL ONLY

In the sixteenth century, Italian mathematicians learned to solve cubic equations. Their formula produced something unexpected. It sometimes asked for the square root of a negative number, even when the equation had three perfectly ordinary answers. No number they knew could give a negative square. Yet if they wrote the impossible quantity down and kept calculating, it disappeared before the end, and the answers were correct.

This chapter begins with that impossible quantity. We define a number i with $i^2 = -1$. No real number does this, so i is new. For a long time many mathematicians distrusted it, and the name it was given, "imaginary", records that distrust. The name was unfair. Once i is allowed, every non-constant polynomial equation has a solution, a result now called the **Fundamental Theorem of Algebra**.

These new numbers are also geometric. Each one is a point in a plane, with the real numbers along one axis. Adding two of them moves a point across the plane. Multiplying two of them rotates and stretches. Multiplying by i rotates by a quarter-turn. In ordinary coordinates a rotation takes several steps; here it is one multiplication. This is one reason complex numbers are used widely in physics and engineering.

By the end of this chapter you will be able to write a complex number in three forms: $a + bi$, $r \operatorname{cis} \theta$ and $re^{i\theta}$. You will multiply and divide in each form, find powers with De Moivre's theorem, and find the nth roots of a complex number, including the roots of unity. You will also use complex roots to factorise polynomials and to prove trigonometric identities.

Three forms may seem like more than you need. Each one makes a different calculation easy. Adding is simplest in $a + bi$ form. A power that would fill a page of working in Cartesian form is immediate in polar form. Choosing the form before you start is a skill this chapter practises throughout.

We start with that algebra: numbers of the form $a + bi$.

17.1 Cartesian Form

Everything starts from a single definition that no real number satisfies.

> DEF The **imaginary unit** i satisfies $i^2 = -1$. A **complex number** is any $z = a + bi$ with $a, b \in \mathbb{R}$; $a = \mathrm{Re}(z)$ is the **real part** and $b = \mathrm{Im}(z)$ the **imaginary part**. Of these the booklet prints only the form $z = a + bi$.

Complex numbers are added, subtracted and multiplied exactly like binomials in i. There is one extra rule: replace i^2 by -1 wherever it appears.

That one rule is enough to find every power of i, because the powers repeat in a cycle of four:

$$i^1 = i, \qquad i^2 = -1, \qquad i^3 = i^2 \cdot i = -i, \qquad i^4 = (i^2)^2 = 1,$$

and $i^5 = i^4 \cdot i = i$ starts the cycle again. To simplify any power, divide the exponent by 4 and keep only the remainder: $i^{2026} = i^{4(506)+2} = i^2 = -1$.

Worked through. Given $z = 2 + i$ and $w = 5 - 3i$, adding is done part by part: $z + w = (2 + 5) + (1 - 3)i = 7 - 2i$. Multiplying, expand and then use $i^2 = -1$:

$$zw = (2 + i)(5 - 3i) = 10 - 6i + 5i - 3i^2 = 10 - i + 3 = 13 - i.$$

The whole of complex arithmetic is ordinary algebra plus the one substitution $i^2 = -1$.

17.1.1 The conjugate and division

Every complex number has a mirror image across the real axis, called its **conjugate**. The conjugate is what makes division possible.

> DEF The **complex conjugate** of $z = a + bi$ is $z^* = a - bi$.

The conjugate matters because a number times its own conjugate is always real: $zz^* = (a + bi)(a - bi) = a^2 + b^2$. This is exactly the tool for division, in the same way that a surd is rationalised. To divide, multiply top and bottom by the conjugate of the denominator, turning the denominator real.

Worked through. To find $\dfrac{3 + 2i}{1 - i}$, multiply numerator and denominator by the conjugate $1 + i$:

$$\frac{3 + 2i}{1 - i} \cdot \frac{1 + i}{1 + i} = \frac{3 + 3i + 2i + 2i^2}{1^2 + 1^2} = \frac{3 + 5i - 2}{2} = \frac{1 + 5i}{2} = \tfrac{1}{2} + \tfrac{5}{2}i.$$

Multiplying by the conjugate removes i from the denominator every time; the answer is then read off as real and imaginary parts.

One more Cartesian tool: two complex numbers are equal exactly when their real parts agree *and* their imaginary parts agree. A single complex equation therefore carries two real equations at once, and **equating parts** solves for unknowns.

EXAMPLE 17.1

Find the complex number z satisfying $z + 2z^* = 9 + 2i$.
Write $z = a + bi$, so $z^* = a - bi$:

$$(a + bi) + 2(a - bi) = 3a - bi = 9 + 2i.$$

Equate real parts: $3a = 9 \implies a = 3$. Equate imaginary parts: $-b = 2 \implies b = -2$. So $z = 3 - 2i$. One equation was enough for two unknowns, because the real and imaginary parts each give an equation.

17.1.2 The Argand diagram

A complex number carries two real numbers, so it is naturally a *point* in a plane: the horizontal axis for the real part, the vertical for the imaginary. This picture, the **Argand diagram**, is what makes the subject geometric.

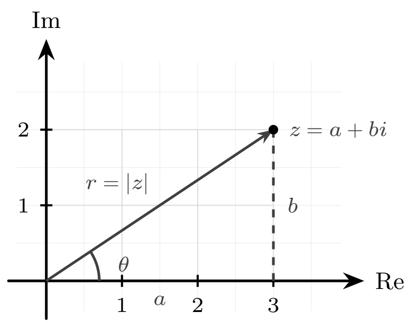

Figure 17.1 A complex number $z = a + bi$ drawn as the point (a, b) of the Argand diagram. Its distance from the origin is the modulus r, and the angle its arrow makes with the positive real axis is the argument θ.

The picture immediately explains the arithmetic. **Addition is vector addition**: to add w to z, place w's arrow tail-to-tip on z's, exactly as in Ch. 9, so $z + w$ is the far corner of the parallelogram the two arrows span. And **conjugation is reflection**: z^* is z flipped across the real axis. This is the picture behind $zz^* = a^2 + b^2$ being real, a fact already checked algebraically above.

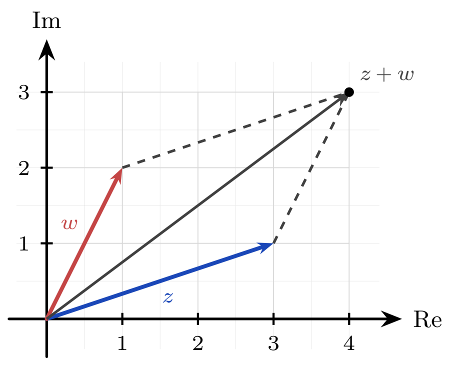

Figure 17.2 Addition is vector addition. Placing w's arrow tail-to-tip on z's puts $z + w$ at the far corner of the parallelogram the two arrows span.

Conjugation is the second picture. Later in the chapter it explains why the non-real roots of a polynomial with real coefficients come in pairs placed symmetrically about the real axis.

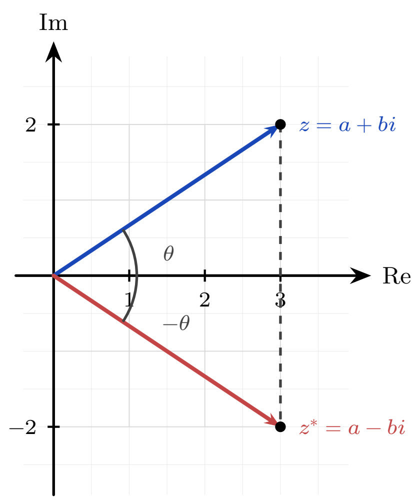

Figure 17.3 The conjugate is a reflection in the real axis. The two numbers share a modulus, and their arguments are equal and opposite. Because conjugating respects addition and multiplication and leaves every real number alone, the roots of a polynomial with real coefficients appear in mirror-image pairs. See §17.4.3.

There are two natural quantities to attach to a point in a plane: how far it is from the origin, and in what direction. These are the modulus and the argument, and they are the subject of the next section.

YOUR TURN 17.1 Cartesian Form

Arithmetic in Cartesian form

1. Evaluate each expression, giving your answer in the form $a + bi$.

(a) $(3 + 2i) + (1 - 4i)$
(b) $(5 - i) - (2 + 3i)$
(c) $(2 + 3i)(1 - i)$
(d) $(2 + i)(2 - i)$
(e) $(1 + 2i)^2(3 - i)$

2. Simplify each power of i.

(a) i^3
(b) i^4
(c) i^{15}
(d) i^{100}

3. Let $z = 7 - 4i$.

(a) Write down $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.
(b) Write down z^*.
(c) Evaluate zz^*.
(d) Evaluate $z + z^*$.

Division

4. Write each quotient in the form $a + bi$.

(a) $\dfrac{4 + 3i}{2 - i}$
(b) $\dfrac{1}{1 + i}$
(c) $\dfrac{2 - 3i}{i}$
(d) $\dfrac{(1 + i)^2}{2 - i}$

Equations involving z^*

5. Find the complex number z in each case.

(a) $z + 3z^* = 8 - 4i$
(b) $2z - z^* = 3 + 6i$
(c) $zz^* + 2z = 12 + 4i$ (there are two answers)

The Argand diagram

6. Let $z = 2 + i$ and $w = -1 + 3i$.

(a) Plot z, w and $z + w$ on an Argand diagram, and state the shape formed by the points 0, z, $z + w$ and w.
(b) Plot z^* and w^*, and describe the single transformation that maps each number to its conjugate.
(c) Find $z - w$, and describe how it appears on the diagram in relation to the points z and w.

Concept check

7. A student writes $(3 + 2i)^2 = 9 + 4i^2 = 5$. Identify the error, and find $(3 + 2i)^2$ correctly.

17.2 Modulus and Argument

The distance of z from the origin is its **modulus** $|z|$, straight from Pythagoras; the angle its arrow makes with the positive real axis is its **argument** $\arg(z)$, measured anticlockwise.

KEY For $z = a + bi$: $\quad |z| = \sqrt{a^2+b^2}, \qquad \tan(\arg z) = \dfrac{b}{a}.$

CAUTION Finding the argument needs care: $\tan\theta = \frac{b}{a}$ alone cannot tell which quadrant z is in, because tan repeats every π. Always sketch z on the Argand diagram first, then choose the angle in the correct quadrant, conventionally in the range $-\pi < \theta \le \pi$ (the **principal argument**).

The danger is easiest to see with an example. The numbers $1+i$ and $-1-i$ both give $\frac{b}{a} = 1$, so a calculator returns $\arctan 1 = \frac{\pi}{4}$ for both. Only one of them is correct.

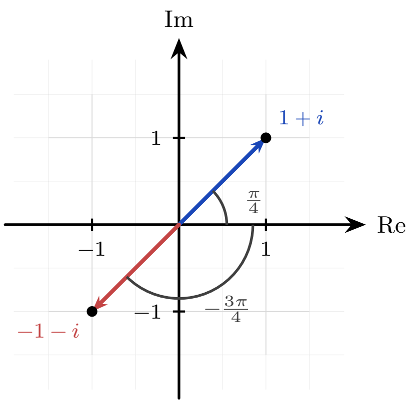

Figure 17.4 Both numbers give $\frac{b}{a} = 1$, so both return $\arctan 1 = \frac{\pi}{4}$ from a calculator. The sketch settles it: $1+i$ lies in the first quadrant and has argument $\frac{\pi}{4}$, while $-1-i$ lies in the third and has principal argument $\frac{\pi}{4} - \pi = -\frac{3\pi}{4}$.

The modulus also measures the distance between two points, not only the distance from the origin. Since $z - w$ is the vector running from w to z Ch. 9, its length $|z-w|$ is the distance between the points representing z and w.

Knowing the modulus r and argument θ is enough to rebuild the number, since $a = r\cos\theta$ and $b = r\sin\theta$. This gives the **modulus–argument** (or polar) form, and the equivalent exponential version, **Euler's form**.

KEY · IB **Polar and Euler form:** $\quad z = r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta = re^{i\theta},$
where $r = |z|$ and $\theta = \arg(z)$.

17.2.1 Euler's form

That $e^{i\theta} = \cos\theta + i\sin\theta$ is unexpected at first sight, and you already have the tools to see why it is true. Take the Maclaurin series for e^x from Ch. 16 and substitute $x = i\theta$, using the cycle of powers of i from §17.1:

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \cdots\right).$$

The even terms, where i has become ± 1, form the cosine series. The odd terms, which still carry an i, form the sine series. So once complex inputs are allowed, the exponential function contains cosine and sine: at $i\theta$ its real part is $\cos\theta$ and its imaginary part is $\sin\theta$. Setting $\theta = \pi$ gives $e^{i\pi} + 1 = 0$, which links e, i, π, 1 and 0 in a single line. It is a calculation, not a coincidence.

Worked through. To write $z = -1 - \sqrt{3}\,i$ in polar and Euler form, with $-\pi < \theta \leq \pi$, start with the modulus. It is $r = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$. The point $\left(-1, -\sqrt{3}\right)$ lies in the third quadrant, where $\tan\theta = \sqrt{3}$ gives a reference angle of $\frac{\pi}{3}$, so the principal argument is $\theta = \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$. Hence

$$z = 2\left(\cos(-\tfrac{2\pi}{3}) + i\sin(-\tfrac{2\pi}{3})\right) = 2\operatorname{cis}(-\tfrac{2\pi}{3}) = 2e^{-2i\pi/3}.$$

Sketch first, then read off the quadrant. Taking $\arctan$ without checking the quadrant gives the wrong angle in the second or third quadrant.

The reverse direction is easier, because no quadrant decision is needed: substitute the angle and evaluate.

EXAMPLE 17.2

Write (a) $4\operatorname{cis}\dfrac{2\pi}{3}$ and (b) $2e^{-i\pi/2}$ in Cartesian form.

(a) Expand the cis and evaluate the two exact values:

$$4\left(\cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3}\right) = 4\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right) = -2 + 2\sqrt{3}\,i.$$

(b) Euler form is the same thing written differently, so read $r = 2$ and $\theta = -\frac{\pi}{2}$:

$$2\left(\cos(-\tfrac{\pi}{2}) + i\sin(-\tfrac{\pi}{2})\right) = 2(0 - i) = -2i.$$

Going from polar to Cartesian is always this direct. It is the other direction that needs the sketch.

YOUR TURN 17.2 Modulus and Argument

Modulus and argument

8. Find the modulus of each number.

(a) $3+4i$
(b) $5-12i$
(c) $1+i$
(d) $-2i$

9. Find the principal argument of each number, with $-\pi < \theta \leq \pi$.

(a) $1+i$
(b) i
(c) -3
(d) $-1-i$
(e) $-\sqrt{3}+i$
(f) $-4-4\sqrt{3}\,i$

Converting between forms

10. Write each number in modulus–argument form and in Euler form.

(a) $2i$
(b) -3
(c) $1-i$
(d) $-1+\sqrt{3}\,i$

11. Write each number in Cartesian form.

(a) $4\,\mathrm{cis}\,\frac{\pi}{3}$
(b) $2e^{-i\pi/4}$
(c) $3\,\mathrm{cis}\,\pi$
(d) $e^{i\pi/2}$

Distance on the Argand diagram

12. Let $z = 3+4i$ and $w = -1+2i$.

(a) Find $|z-w|$, giving an exact answer.
(b) State what $|z-w|$ measures on the Argand diagram.
(c) Verify that $zz^* = |z|^2$.

Concept check

13. A student writes $\arg\left(-1-\sqrt{3}\,i\right) = \arctan\dfrac{-\sqrt{3}}{-1} = \arctan\sqrt{3} = \dfrac{\pi}{3}$. Identify the error, and find the principal argument.

17.3 Products, Quotients and De Moivre's Theorem

Polar form needs extra notation, and it repays that cost by making multiplication easy. When two complex numbers are multiplied, their moduli multiply and their arguments *add*, which is just the law of exponents applied to $re^{i\theta}$.

KEY $|zw| = |z|\,|w|$ and $\arg(zw) = \arg z + \arg w$; $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ and $\arg\left(\dfrac{z}{w}\right) = \arg z - \arg w$.

With principal arguments these rules hold up to a multiple of 2π: if the sum or difference falls outside $-\pi < \theta \leq \pi$, add or subtract 2π. Read geometrically, multiplying by a complex number *rotates* by its argument and *scales* by its modulus. So multiplying by $i = 1 \cdot e^{i\pi/2}$, whose modulus is 1, is a quarter-turn anticlockwise with no stretching at all.

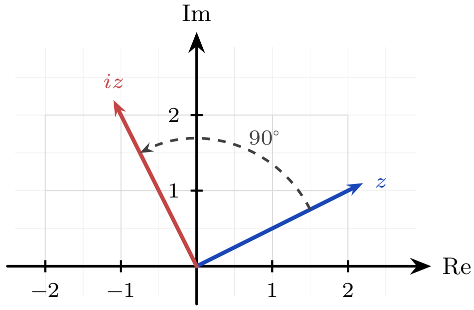

Figure 17.5 Multiplying by i turns a complex number through a quarter-turn anticlockwise about the origin, leaving its modulus unchanged.

The quotient rule with $z = 1$ gives the reciprocal: for $r > 0$,

$$\frac{1}{r\,\mathrm{cis}\,\theta} = \frac{1}{r}\,\mathrm{cis}(-\theta).$$

Taking the reciprocal replaces the modulus by its reciprocal and reverses the argument, so for $|z| = 1$ the reciprocal is the conjugate z^*.

Worked through. Take $z = 6\,\mathrm{cis}\,\frac{2\pi}{3}$ and $w = 3\,\mathrm{cis}\,\frac{\pi}{6}$. For the product, multiply the moduli and add the arguments:

$$zw = (6)(3)\,\mathrm{cis}\left(\tfrac{2\pi}{3} + \tfrac{\pi}{6}\right) = 18\,\mathrm{cis}\,\tfrac{5\pi}{6}.$$

For the quotient, divide the moduli and subtract the arguments:

$$\frac{z}{w} = \frac{6}{3}\,\mathrm{cis}\left(\tfrac{2\pi}{3} - \tfrac{\pi}{6}\right) = 2\,\mathrm{cis}\,\tfrac{\pi}{2} = 2i.$$

Compare this with doing the same multiplication in Cartesian form, which would need four products and a collection of terms. Once the numbers are in polar form the work is one multiplication and one addition.

If multiplying adds arguments, then raising to a power *multiplies* the argument. This is **De Moivre's theorem**, and the roots, the unit-circle results and the trigonometric identities in the rest of the chapter all follow from it.

KEY · IB **De Moivre's theorem:**
$\left(r(\cos\theta + i\sin\theta)\right)^n = r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta} = r^n \operatorname{cis} n\theta.$

The syllabus requires the theorem for $n \in \mathbb{Z}$ and extends it to *rational* exponents. Past the integers the statement has to be read more carefully: a power like $z^{1/n}$ is multi-valued, so $\sqrt[n]{r}\operatorname{cis}\frac{\theta}{n}$ is one of its n values rather than the whole answer. That is exactly the story of the nth roots in the next section. For positive integers the theorem is proved by the induction of Ch. 3. The inductive step is one multiplication: assuming $(r\operatorname{cis}\theta)^k = r^k \operatorname{cis} k\theta$, multiplying both sides once more by $r\operatorname{cis}\theta$ scales the modulus to r^{k+1} and, by the compound-angle identities of Ch. 8, adds θ to the argument to give $(k+1)\theta$. Every step is that same identity applied once, and you carry the induction out in full in End-of-Chapter Problem 22.

EXAMPLE 17.3

The theorem is claimed for every integer exponent. Taking the case $n \in \mathbb{Z}^+$ as proved, show that it also holds for every negative integer n, for $r > 0$.

Start with the reciprocal of a number of modulus 1. Multiplying adds the arguments, so

$$\operatorname{cis}\phi \cdot \operatorname{cis}(-\phi) = \operatorname{cis}(\phi - \phi) = \operatorname{cis} 0 = 1, \qquad \text{hence} \qquad \frac{1}{\operatorname{cis}\phi} = \operatorname{cis}(-\phi).$$

Dividing by a number of modulus 1 simply reverses its angle.

Now let n be a negative integer, and write $n = -m$ with $m \in \mathbb{Z}^+$. Applying the positive case to m,

$$\left(r\operatorname{cis}\theta\right)^n = \frac{1}{\left(r\operatorname{cis}\theta\right)^m} = \frac{1}{r^m \operatorname{cis}(m\theta)} = r^{-m}\operatorname{cis}(-m\theta) = r^n \operatorname{cis}(n\theta). \quad \blacksquare$$

The exponent $n = 0$ needs no work either, since $z^0 = 1 = r^0 \operatorname{cis} 0$. So one reciprocal carries the theorem from the positive integers to the whole of $\mathbb{Z}$, which is the range the syllabus asks for.

Raising a complex number to a high power means expanding a bracket again and again in Cartesian form. In polar form there is nothing to expand: raise the modulus to the power, multiply the angle by the power, and the work is finished.

EXAMPLE 17.4

Find $(1+i)^8$.

In polar form $1 + i = \sqrt{2}\,\operatorname{cis}\frac{\pi}{4}$. By De Moivre,

$$(1+i)^8 = (\sqrt{2})^8\,\operatorname{cis}(8 \cdot \tfrac{\pi}{4}) = 16\,\operatorname{cis}(2\pi) = 16.$$

Seven repeated multiplications reduce to one power of the modulus and one multiple of the angle.

Draw the powers and you can watch the theorem work. Each multiplication by $1+i$ turns the arrow 45° anticlockwise and stretches it by $\sqrt{2}$, so the powers lie on a spiral that winds outward from the origin:

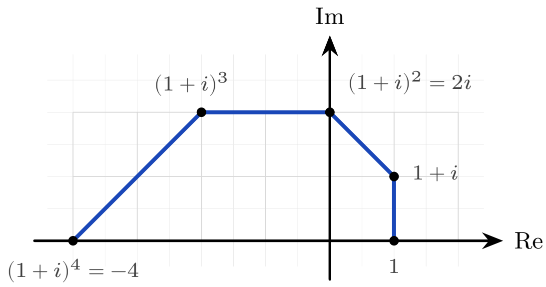

Figure 17.6 The powers $(1+i)^0$ through $(1+i)^4$. Each multiplication turns the arrow through 45° and stretches it by $\sqrt{2}$, so the powers lie on a spiral winding outward from the origin.

Four more steps continue the spiral through the lower half-plane and back to the positive real axis at $(1+i)^8 = 16$: a full turn of $8 \times 45^\circ = 360^\circ$, with the modulus grown to $(\sqrt{2})^8 = 16$. The worked example above is this picture written in algebra.

17.3.1 Numbers on the unit circle

One case of De Moivre is used often enough to state separately. If $|z| = 1$ then $z = \operatorname{cis}\theta$, and by the reciprocal result above, $\dfrac{1}{z} = \operatorname{cis}(-\theta) = z^*$. Add the two and the imaginary parts cancel while the real parts double; subtract them and the opposite happens.

KEY For $z = \operatorname{cis}\theta$ and any $n \in \mathbb{Z}$:

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad z^n - \frac{1}{z^n} = 2i\sin n\theta.$$

Read from left to right this converts powers into multiple angles, and that is what makes it useful: an expression like $\cos^6\theta$, awkward to integrate as it stands, becomes a sum of $\cos 2\theta$, $\cos 4\theta$ and $\cos 6\theta$ terms that can be integrated directly. That construction is §17.5.

EXAMPLE 17.5

Let $z = \operatorname{cis}\theta$. Expand $\left(z + \dfrac{1}{z}\right)\left(z - \dfrac{1}{z}\right)$ in two ways, and hence express $\sin\theta\cos\theta$ in terms of $\sin 2\theta$.

Taking $n = 1$ in the two formulas, $z + \dfrac{1}{z} = 2\cos\theta$ and $z - \dfrac{1}{z} = 2i\sin\theta$, so the product is

$$\left(z + \frac{1}{z}\right)\left(z - \frac{1}{z}\right) = (2\cos\theta)(2i\sin\theta) = 4i\sin\theta\cos\theta.$$

Expanding algebraically instead, the two cross terms cancel:

$$\left(z + \frac{1}{z}\right)\left(z - \frac{1}{z}\right) = z^2 - 1 + 1 - \frac{1}{z^2} = z^2 - \frac{1}{z^2} = 2i\sin 2\theta,$$

using the second formula again with $n = 2$. The two expressions are equal, so

$$4i \sin\theta \cos\theta = 2i \sin 2\theta \implies \sin\theta\cos\theta = \frac{\sin 2\theta}{2}.$$

This is the double-angle identity for the sine, from Ch. 8, now obtained without using the compound-angle formula directly. Evaluating one product in two ways is the technique of §17.5.1.

YOUR TURN 17.3 Products, Quotients and De Moivre's Theorem

Products and quotients

14. Evaluate each expression, giving your answer in modulus–argument form.

(a) $\left(2\operatorname{cis}\frac{\pi}{6}\right)\left(3\operatorname{cis}\frac{\pi}{3}\right)$ (b) $\dfrac{6\operatorname{cis}\frac{2\pi}{3}}{2\operatorname{cis}\frac{\pi}{3}}$

15. Given $|z| = 3$, $|w| = 4$, $\arg z = \frac{\pi}{4}$ and $\arg w = \frac{\pi}{6}$, write down

(a) $|zw|$ (b) $\arg(zw)$

(c) $\left|\dfrac{z}{w}\right|$ (d) $\arg\left(\dfrac{z}{w}\right)$

16. Give each answer in modulus–argument form, with the principal argument $-\pi < \theta \leq \pi$.

(a) Find zw, where $z = 2\operatorname{cis}\frac{5\pi}{6}$ and $w = 3\operatorname{cis}\frac{\pi}{3}$.

(b) Find $\dfrac{z}{w}$, where $z = 4\operatorname{cis}\left(-\frac{2\pi}{3}\right)$ and $w = 2\operatorname{cis}\frac{\pi}{2}$.

De Moivre's theorem

17. Use De Moivre's theorem for each part.

(a) Write $(\cos\theta + i\sin\theta)^4$ in the form $\cos n\theta + i\sin n\theta$.

(b) Evaluate $\left(\operatorname{cis}\frac{\pi}{5}\right)^5$, in modulus–argument form.

(c) Evaluate $\left(2\operatorname{cis}\frac{\pi}{6}\right)^3$, in modulus–argument form.

(d) Evaluate $(1+i)^6$.

(e) Evaluate $\left(1-\sqrt{3}\,i\right)^5$.

Negative powers and reciprocals

18.

(a) Find $\left(2\operatorname{cis}\frac{\pi}{6}\right)^{-2}$ in the form $a + bi$.

(b) Find $(1-i)^{-4}$.

(c) Write $\dfrac{1}{\sqrt{3}+i}$ in modulus–argument form.

Multiplying as a rotation

19.

(a) Let $z = 3 + 2i$. Find iz, and describe the transformation of the Argand diagram that maps z to iz.

(b) The point $4 + 2i$ is rotated anticlockwise about the origin through $\frac{\pi}{3}$. Find its image in the form $a + bi$.

(c) $OABC$ is a square on the Argand diagram, with vertices labelled anticlockwise, O at the origin and A at $2 + i$. Find the numbers at C and at B.

Numbers on the unit circle

20. Let $z = \operatorname{cis}\theta$. Write each expression in terms of a single cosine or sine.

(a) $z + \dfrac{1}{z}$

(b) $z^3 + \dfrac{1}{z^3}$

(c) $z^2 - \dfrac{1}{z^2}$

Concept check

21. A student uses De Moivre's theorem and writes $(1 + i)^4 = \operatorname{cis}\left(4 \times \frac{\pi}{4}\right) = \operatorname{cis}\pi = -1$. Identify the error, and find $(1 + i)^4$.

17.4 Roots of Complex Numbers

Solving $z^3 = 125$ over the real numbers gives one answer. Over the complex numbers it gives three, and they are not scattered: they sit at equal spacing on a circle. The first part of this section is built on that picture.

De Moivre's theorem, used in reverse, finds these roots. A nonzero complex number has exactly n distinct nth roots. They all share one modulus, and their arguments are equally spaced. To find them, write the number in polar form, take the nth root of the modulus, divide the argument by n, then step around by $\frac{2\pi}{n}$ until all n are collected.

KEY The nth roots of $z = r\operatorname{cis}\theta$ are $\sqrt[n]{r}\,\operatorname{cis}\left(\dfrac{\theta + 2\pi k}{n}\right)$, for $k = 0, 1, \ldots, n-1$.

Because the roots share a modulus $\sqrt[n]{r}$ and their arguments differ by equal steps of $\frac{2\pi}{n}$, for $n \geq 3$ they sit at the vertices of a **regular n-gon** centred at the origin (for $n = 2$, at the two ends of a diameter). One root therefore locates all the others: find any single root, and the rest follow by turning it through $\frac{2\pi}{n}$ at a time.

KEY To solve $z^n = w$:

1. Write w in polar form, $w = r \operatorname{cis} \theta$.
2. The modulus of every root is $\sqrt[n]{r}$.
3. Take $\dfrac{\theta}{n}$ as the first argument, then add $\dfrac{2\pi}{n}$ repeatedly until you have n roots.
4. Convert back to Cartesian form only if the question asks for it.

Worked through. To find the three cube roots of 125, write $125 = 125 \operatorname{cis} 0$. The modulus of each root is $\sqrt[3]{125} = 5$, and the arguments are $\frac{0+2\pi k}{3}$ for $k = 0, 1, 2$:

$$5 \operatorname{cis} 0 = 5, \qquad 5 \operatorname{cis} \tfrac{2\pi}{3} = -\frac{5}{2} + \frac{5\sqrt{3}}{2}i, \qquad 5 \operatorname{cis} \tfrac{4\pi}{3} = -\frac{5}{2} - \frac{5\sqrt{3}}{2}i.$$

The three roots form an equilateral triangle on the circle of radius 5, one real and a conjugate pair. Only the first argument needed any thought; the other two came from adding $\frac{2\pi}{3}$.

That calculation hid one step, because 125 has argument 0 and dividing 0 by 3 changes nothing. When the number has a non-zero argument, the division does real work.

EXAMPLE 17.6

Find the three cube roots of $-27i$, giving each in polar form and in Cartesian form.

First write $-27i$ in polar form. It lies on the negative imaginary axis, so its modulus is 27 and its argument is $-\frac{\pi}{2}$:

$$-27i = 27 \operatorname{cis}\left(-\tfrac{\pi}{2}\right).$$

Each root has modulus $\sqrt[3]{27} = 3$. The first argument is the given argument divided by 3:

$$\frac{-\pi/2}{3} = -\frac{\pi}{6}.$$

The others follow by adding $\frac{2\pi}{3}$ each time:

$$-\frac{\pi}{6}, \qquad -\frac{\pi}{6} + \frac{2\pi}{3} = \frac{\pi}{2}, \qquad \frac{\pi}{2} + \frac{2\pi}{3} = \frac{7\pi}{6} \equiv -\frac{5\pi}{6}.$$

The third has been reduced by 2π to bring it back into $-\pi < \theta \le \pi$. In polar form the roots are $3 \operatorname{cis}\left(-\frac{\pi}{6}\right)$, $3 \operatorname{cis} \frac{\pi}{2}$ and $3 \operatorname{cis}\left(-\frac{5\pi}{6}\right)$, and converting each,

$$\frac{3\sqrt{3}}{2} - \frac{3}{2}i, \qquad 3i, \qquad -\frac{3\sqrt{3}}{2} - \frac{3}{2}i.$$

Check the easiest one: $(3i)^3 = 27i^3 = -27i$. ✓ Note that adding $\frac{2\pi}{3}$ repeatedly can carry an argument past π; subtract 2π to return it to the principal range if the question asks for principal arguments.

17.4.1 Roots of Unity

One case is used often enough to study separately: $w = 1$. Solving $z^n = 1$ gives the **roots of unity**. These are n points on the *unit* circle. One of them is 1, and together they sit at the vertices of a regular n-gon.

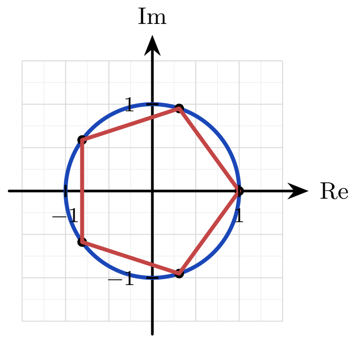

Figure 17.7 The five fifth roots of unity, at the vertices of a regular pentagon inscribed in the unit circle with one vertex at 1.

One root generates all the others. Write $\omega = \operatorname{cis}\dfrac{2\pi}{n}$ for the first root anticlockwise from 1. Then $\omega^k = \operatorname{cis}\dfrac{2\pi k}{n}$, which is the kth root round the circle. Every power of ω is again a root, because $\left(\omega^k\right)^n = \left(\omega^n\right)^k = 1^k = 1$. The n roots are therefore

$$1,\ \omega,\ \omega^2,\ \dots,\ \omega^{n-1},$$

one number and its powers, stepping round the polygon and returning to the start.

The case $n = 3$ is worth memorising. Converting each root to Cartesian form, the cube roots of unity are 1, $\omega = \operatorname{cis}\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega^2 = \operatorname{cis}\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$: a real root and a conjugate pair on the unit circle.

For $n \geq 2$ they also **sum to zero**. There are two ways to see this. Geometrically, the n arrows point in n equally spaced directions, so they cancel. Algebraically, the geometric series of Ch. 2 gives the result directly, since $\omega \neq 1$:

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = \frac{\omega^n - 1}{\omega - 1} = \frac{1 - 1}{\omega - 1} = 0.$$

The same conclusion follows from Ch. 6. The polynomial $z^n - 1$ has no z^{n-1} term, so the sum of its roots is zero. For $n = 3$ this gives the identity $1 + \omega + \omega^2 = 0$, which the next example uses twice.

EXAMPLE 17.7

Let $\omega = \operatorname{cis}\dfrac{2\pi}{3}$, so that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$. Simplify (a) $\omega^{10} + \omega^{20}$, and (b) $(1 + \omega)(1 + \omega^2)$.

(a) A power of ω is reduced by dividing the exponent by 3 and keeping the remainder, since

$\omega^3 = 1$. Here $10 = 3(3) + 1$ and $20 = 3(6) + 2$, so

$$\omega^{10} + \omega^{20} = \omega + \omega^2 = -1,$$

the last step because $1 + \omega + \omega^2 = 0$.
(b) Expand, and use $\omega^3 = 1$ again:

$$(1 + \omega)(1 + \omega^2) = 1 + \omega^2 + \omega + \omega^3 = (1 + \omega + \omega^2) + 1 = 0 + 1 = 1.$$

Both parts turned on two facts: $\omega^3 = 1$ reduces any power of ω, and $1 + \omega + \omega^2 = 0$ removes what is left. Apply them in that order.

These evenly spaced points have a large practical use. The *discrete Fourier transform* separates a signal, such as a sound, into the pure frequencies it is made of. It does this by multiplying the n samples by powers of these n points on the circle and adding the results. A fast method of computing it, the fast Fourier transform, uses the symmetry of the polygon to cut the work, and it is one of the methods that let a phone process sound as it arrives.

17.4.2 Square Roots Without Polar Form

Polar form finds roots of every order, but it has one drawback. It needs the argument, and unless the angle is one of the standard ones, $\arctan$ returns a decimal. The answer is then no longer exact. For *square* roots there is a second route that stays in Cartesian form and keeps the surds intact. Assume the root is $a + bi$, square it, and equate real and imaginary parts as in §17.1.

Worked through. To find the square roots of $7 + 24i$ in the form $a + bi$, let $(a + bi)^2 = 7 + 24i$. Expanding the left side gives $a^2 - b^2 + 2abi$, so equating parts,

$$a^2 - b^2 = 7 \qquad \text{and} \qquad 2ab = 24.$$

The second gives $b = \dfrac{12}{a}$. Substituting into the first,

$$a^2 - \frac{144}{a^2} = 7 \implies a^4 - 7a^2 - 144 = 0 \implies (a^2 - 16)(a^2 + 9) = 0.$$

Since a is real, $a^2 = 16$, so $a = \pm 4$ and correspondingly $b = \pm 3$. The square roots are $4 + 3i$ and $-4 - 3i$, that is $\pm(4 + 3i)$.

Two features of that calculation repeat every time. When the imaginary part is non-zero, the substitution always produces a **quadratic in a^2** with one positive and one negative solution, and the negative one is rejected, because a is real. The two answers always differ by a sign, since square roots lie at opposite ends of a diameter, half a turn apart.

CAUTION Use Cartesian equating for square roots when the answer should be exact. For cube roots and higher, go back to polar form: equating parts would give a cubic or an equation of even higher degree, while De Moivre handles roots of any order directly.

17.4.3 Complex Roots of Polynomials

Every nth root above was a solution of $z^n = w$, which is a polynomial equation of degree n with exactly n solutions. That is not a coincidence of this one family. This is the **Fundamental Theorem of Algebra**: every polynomial of degree $n \geq 1$ has exactly n roots in the complex numbers, counted with multiplicity. The roots of a polynomial need not lie on the real line, but none of them can lie outside the complex plane.

The simplest case has already appeared, in a different form. In Ch. 4, a quadratic with negative discriminant had "no solutions." It always had two roots; they were simply not on the real line.

Worked through. To solve $z^2 - 4z + 13 = 0$, use the quadratic formula, which works unchanged:

$$z = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i.$$

The negative discriminant simply produces an i: the two roots are $2 + 3i$ and $2 - 3i$, a conjugate pair placed symmetrically above and below the real axis.

That symmetry is no accident. When a polynomial has *real* coefficients, its non-real roots always come in conjugate pairs.

KEY **Conjugate root theorem.** If a polynomial has real coefficients and $z = a + bi$ is a root, then $z^* = a - bi$ is also a root.

This is why a cubic with real coefficients always has at least one real root: its three roots are either all real, or one real root plus a conjugate pair. A non-real root never appears alone in a polynomial with real coefficients.

To use the theorem, turn the pair into a real factor. A conjugate pair $a \pm bi$ gives the real quadratic factor $(z - a)^2 + b^2 = z^2 - 2az + (a^2 + b^2)$. Divide the polynomial by this factor, or compare coefficients, to find the remaining roots.

EXAMPLE 17.8

Given that $2 + i$ is a root of a polynomial with real coefficients, state another root and find a real quadratic factor.

By the conjugate root theorem, $2 - i$ is also a root. The two together give a quadratic factor with real coefficients:

$$(z - (2 + i))(z - (2 - i)) = (z - 2)^2 - i^2 = z^2 - 4z + 5.$$

This is the real quadratic factor $(z - a)^2 + b^2$ with $a = 2$ and $b = 1$; dividing it out of the polynomial gives the remaining roots.

YOUR TURN 17.4 Roots of Complex Numbers

How many roots, and where

22.

(a) Write down a square root of -9.
(b) State the modulus of each cube root of 27.
(c) State how many distinct fifth roots a non-zero complex number has.
(d) State the shape formed on the Argand diagram by the nth roots $(n \geq 3)$ of a non-zero complex number.

Roots of unity

23.

(a) State the three cube roots of unity, in Cartesian form.
(b) State the four fourth roots of unity.
(c) State the six sixth roots of unity, in modulus–argument form.
(d) Let ω be a cube root of unity with $\omega \neq 1$. Show that $1 + \omega + \omega^2 = 0$.

24. Let $\omega = \operatorname{cis} \frac{2\pi}{3}$, so that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.

(a) Simplify $(1 - \omega)(1 - \omega^2)$.
(b) Show that $\left(1 + \omega^2\right)^4 = \omega$.

Finding nth roots

25. Find, in modulus–argument form with principal arguments,

(a) the three cube roots of 8;
(b) the four fourth roots of 16;
(c) the three cube roots of i.

26. Find the square roots of each number in the form $a + bi$ by equating real and imaginary parts.

(a) $5 + 12i$
(b) $-8 + 6i$
(c) $-5 + 12i$

Polynomials with real coefficients

27. Solve each equation.

(a) $z^2 + 9 = 0$
(b) $z^2 - 2z + 5 = 0$
(c) $z^2 + 2z + 2 = 0$
(d) $2z^2 - 2z + 5 = 0$

28.

(a) Given that $3 + i$ is a root of a polynomial with real coefficients, state another root.
(b) Write down a quadratic with real coefficients whose roots are $2i$ and $-2i$.
(c) Write down a quadratic with real coefficients whose roots are $-2 + 5i$ and $-2 - 5i$.

29.

(a) The equation $z^2 + bz + c = 0$ has real coefficients and $3 - 2i$ as a root. Find b and c.
(b) Explain why a cubic with real coefficients must have at least one real root.

30.

(a) Given that $1 - i$ is a root of $z^3 - 4z^2 + 6z - 4 = 0$, find the other two roots.
(b) Given that $2i$ is a root of $z^4 - 2z^3 + 5z^2 - 8z + 4 = 0$, solve the equation.
(c) The equation $z^3 + az^2 + bz - 26 = 0$, where $a, b \in \mathbb{R}$, has $2 - 3i$ as a root. Find a and b, and the real root.

Concept check

31. A student solves $z^3 = 8i$ and writes $z = 2\,\mathrm{cis}\left(\frac{\pi}{6} + 2k\pi\right)$ for $k = 0, 1, 2$. Explain why this gives only one root, and find all three roots in modulus–argument form, with principal arguments.

32. A student says: "*i is a root of $z^2 - 3iz - 2 = 0$, so $-i$ is also a root, because complex roots come in conjugate pairs.*" Show that $-i$ is not a root, and state the condition the student has overlooked.

17.5 Trigonometric Identities from De Moivre

The compound-angle rules of Ch. 8 give $\cos 2\theta$ and $\sin 2\theta$ without much work. For $\cos 3\theta$ they take considerably more. Each multiple angle has to be built from the one before it, so the work grows with every step. One expansion with complex numbers gives any multiple angle at once, and it gives the sine identity and the cosine identity together.

The method comes from one observation. De Moivre's theorem says that $(\cos\theta + i\sin\theta)^n$ and $\cos n\theta + i\sin n\theta$ are the same number. So one number has been written in two ways. Expand the first way with the binomial theorem of Ch. 2. The two ways must then match in both parts: the real part of one equals the real part of the other, and the imaginary parts match in the same way.

KEY To find an identity for $\cos n\theta$ or $\sin n\theta$:

1. Expand $(\cos\theta + i\sin\theta)^n$ by the binomial theorem.
2. Collect the terms into a real part and an imaginary part, using $i^2 = -1$.
3. Equate the real part to $\cos n\theta$ and the imaginary part to $\sin n\theta$.
4. Replace $\sin^2\theta$ by $1 - \cos^2\theta$, or the reverse, to leave one function only.

Each expansion is compared in two parts, so it always produces two identities at once. Divide one identity by the other and you have a third: the imaginary part over the real part is $\tan n\theta$, and the real part over the imaginary part is $\cot n\theta$.

Worked through. To show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, and to find a matching identity for $\sin 3\theta$, write $c = \cos\theta$, $s = \sin\theta$, and expand by the binomial theorem:

$$(c + is)^3 = c^3 + 3c^2(is) + 3c(is)^2 + (is)^3 = (c^3 - 3cs^2) + i(3c^2 s - s^3).$$

By De Moivre this equals $\cos 3\theta + i\sin 3\theta$, so the parts must match separately. Real parts, using $s^2 = 1 - c^2$:

$$\cos 3\theta = c^3 - 3c(1 - c^2) = 4\cos^3\theta - 3\cos\theta.$$

Imaginary parts, using $c^2 = 1 - s^2$:

$$\sin 3\theta = 3(1 - s^2)s - s^3 = 3\sin\theta - 4\sin^3\theta.$$

One expansion gives two identities, and the same method works for any n. Results that took several steps with compound angles in Ch. 8 now follow from a single binomial expansion.

A third identity comes from the same expansion and needs no new work: divide one part by the other.

EXAMPLE 17.9

Use the expansion above to show that $\cot 3\theta = \dfrac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}$.
Take the two parts of the expansion *before* either was rewritten in a single function:

$$\cos 3\theta = c^3 - 3cs^2, \qquad \sin 3\theta = 3c^2 s - s^3.$$

Since $\cot 3\theta$ is the first divided by the second,

$$\cot 3\theta = \frac{c^3 - 3cs^2}{3c^2 s - s^3}.$$

Now divide every term, top and bottom, by s^3. Each division turns a c over an s into a $\cot\theta$:

$$\cot 3\theta = \frac{\dfrac{c^3}{s^3} - \dfrac{3cs^2}{s^3}}{\dfrac{3c^2 s}{s^3} - \dfrac{s^3}{s^3}} = \frac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}. \quad ■$$

Two points make this reliable. Use the parts in their original mixed form, since replacing s^2 by $1-c^2$ first would leave a mixture that no division tidies. And divide by the highest power present, here s^3, so that every term becomes a power of a single ratio.

17.5.1 Powers of Cosine and Sine

The method above turns a multiple angle into powers of $\cos\theta$. The same idea works in the other direction. A power such as $\cos^6\theta$ becomes a sum of multiple angles, and a sum of multiple angles can be integrated directly with the methods of Ch. 15.

The tool is the pair of results from §17.3. For $z = \operatorname{cis}\theta$,

$$z + \frac{1}{z} = 2\cos\theta, \qquad z^n + \frac{1}{z^n} = 2\cos n\theta,$$

and the same expressions with a minus sign give $2i\sin\theta$ and $2i\sin n\theta$. So raise $z + \frac{1}{z}$ to a power and expand it. Each pair of terms then combines into a cosine of a multiple angle.

Worked through. To express $\cos^6\theta$ in terms of $\cos 2\theta$, $\cos 4\theta$ and $\cos 6\theta$, start from $2\cos\theta = z + \dfrac{1}{z}$ and raise both sides to the sixth power, with binomial coefficients $1, 6, 15, 20, 15, 6, 1$:

$$64\cos^6\theta = \left(z + \frac{1}{z}\right)^6 = z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}.$$

Now pair the terms that are reciprocals of each other:

$$64\cos^6\theta = \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20 = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20.$$

Divide by 64:

$$\cos^6\theta = \tfrac{1}{32}\cos 6\theta + \tfrac{3}{16}\cos 4\theta + \tfrac{15}{32}\cos 2\theta + \tfrac{5}{16}.$$

The pairing is what makes the method work. In the expansion, the first term z^6 is the reciprocal of the last term $\frac{1}{z^6}$ and has the same binomial coefficient, and the same holds for the second term and the second from last. Every pair therefore combines into a cosine.

TIP A power of $\sin\theta$ works the same way, starting from $2i\sin\theta = z - \frac{1}{z}$. Take one extra care. Raising i to the power leaves a factor that must be divided out at the end, and an odd power gives sines rather than cosines. Check your answer by putting $\theta = \frac{\pi}{2}$, where $\sin\theta = 1$.

TIP Complex numbers offer several good IA topics. Three directions to explore: iterate

$z \mapsto z^2 + c$ starting from $z = 0$ and investigate which values of c make the sequence escape to infinity (this is how the Mandelbrot set is defined); investigate which regular polygons can be constructed with ruler and compass, a question the Greeks asked, which Gauss advanced at eighteen when he used exactly the roots of unity above to construct the regular 17-gon, and which Wantzel completed in 1837; or measure phase shift in a simple AC circuit and model it with rotating complex numbers.

YOUR TURN 17.5 Trigonometric Identities from De Moivre

Multiple angles as powers

33. Use De Moivre's theorem and the binomial theorem for each part.

(a) Show that $\cos 2\theta = 2\cos^2\theta - 1$.

(b) Find a matching identity for $\sin 2\theta$.

(c) Show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

(d) Hence, or otherwise, express $\tan 2\theta$ in terms of $\tan\theta$.

Powers as multiple angles

34. Let $z = \operatorname{cis}\theta$. Use $z \pm \dfrac{1}{z}$, where $z + \dfrac{1}{z} = 2\cos\theta$ and $z - \dfrac{1}{z} = 2i\sin\theta$.

(a) Express $\cos^2\theta$ in terms of $\cos 2\theta$.

(b) Express $\sin^2\theta$ in terms of $\cos 2\theta$.

(c) Express $\cos^3\theta$ in terms of $\cos 3\theta$ and $\cos\theta$.

35.

(a) Show that $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$.

(b) Express $\cos^5\theta$ in terms of cosines of multiple angles.

Integrating a power of sine or cosine

36.

(a) Use your answer to Question 34(c) to evaluate $\displaystyle\int_0^{\pi/2} \cos^3\theta \, d\theta$.

(b) Use the result of Question 35(a) to find $\displaystyle\int \sin^3\theta \, d\theta$.

Concept check

37. A student uses $z = \operatorname{cis}\theta$ and writes

$$\left(z - \frac{1}{z}\right)^2 = (2i\sin\theta)^2 = 4\sin^2\theta \quad \text{and} \quad \left(z - \frac{1}{z}\right)^2 = z^2 - 2 + \frac{1}{z^2} = 2\cos 2\theta - 2,$$

and concludes that $\sin^2\theta = \frac{1}{2}(\cos 2\theta - 1)$. Show that this result is false by taking $\theta = \frac{\pi}{2}$, identify the error, and correct the result.

Chapter 17 · Complex Numbers

Reflections

TOK Euler's identity $e^{i\pi} + 1 = 0$ has often been called the most beautiful equation in mathematics. What is being judged when a mathematician calls a result beautiful, and is that judgement evidence of anything? Beauty in mathematics is often described as economy, surprise, or unexpected connection, all of which this identity has.

IM The name "imaginary" was meant as a dismissal. Descartes coined it in the seventeenth century for numbers he regarded as fictions, and Leibniz called them "an amphibian between being and non-being." Their rehabilitation was slow and international. The Norwegian-born Danish surveyor Caspar Wessel and the Swiss bookkeeper Jean-Robert Argand independently drew the plane picture that made them concrete. The German mathematician Carl Friedrich Gauss did much to make them respectable: his 1799 proof of the Fundamental Theorem of Algebra was the first to be widely accepted, although it contained a gap that was closed only in 1920, and he popularised the term "complex" itself. Gauss also remarked that these numbers would have met less resistance if they had been called "lateral" rather than imaginary. Does the name of an idea change how readily it is believed?

RW Complex numbers are a standard tool for describing oscillations. An alternating current, a radio wave or a vibration is modelled as a complex number rotating about the origin, and its real part is the signal you measure. Engineers add and scale these "phasors" directly, in place of pages of trigonometry. Quantum mechanics goes further: a particle's state is a complex-valued wave, and i appears explicitly in the Schrödinger equation, which describes how that state changes over time. Phone signal processing, power-grid impedance and quantum computing all use the numbers of this chapter.

EXT De Moivre's theorem connects two families of functions. The identity $e^{i\theta} = \cos\theta + i\sin\theta$ shows that the exponential and trigonometric functions, which look nothing alike on the real line, are closely related once complex inputs are allowed: $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$. Exponential growth and oscillation both come from the one function e^z. Push this further and you reach complex analysis, the study of functions of a complex variable. Such functions turn out to be far more tightly constrained than real ones, and some problems about ordinary integers, such as how the prime numbers are distributed, have been partly answered by studying functions over the complex plane.

End-of-Chapter Problems

1. Given $z = 3 + 4i$ and $w = 1 - 2i$:

(a) find $z + w$; [1]
(b) find zw; [2]
(c) find $\frac{z}{w}$; [2]
(d) find $|zw|$. [1]

2. Express $\dfrac{2+i}{3-i}$ in the form $a + bi$. [4]

3. Simplify

(a) $i^{2027} + i^{2028}$; [2]
(b) i^{-7}. [2]

4. Solve $2z^2 + 2z + 5 = 0$. [4]

5. Find the modulus and the principal argument of

(a) $z = -1 + i$; [2]
(b) $z = \sqrt{3} - i$. [2]

6. The points representing 1, i, -1 and $-i$ are plotted on an Argand diagram. State the shape they form and find its area. [3]

7. Write $z = -\sqrt{3} + 3i$ in modulus–argument form and in Euler form. [4]

8. Given $z = 2 + 2i$, find $|z|$ and $\arg z$, and hence write z in Euler form. [4]

9. Find real numbers x and y such that $(x + yi)(2 - i) = 3 + i$. [4]

10. Find the complex number z satisfying $3z + iz^* = 5 + 7i$. [4]

11.

(a) Express -8 in the form $re^{i\theta}$, with $-\pi < \theta \leq \pi$. [3]
(b) The complex number z has $|z| = 6$ and $\arg z = \frac{3\pi}{4}$. Write z in Cartesian form. [3]

12. Use De Moivre's theorem to evaluate

(a) $(-\sqrt{3}+i)^5$, giving your answer in the form $a+bi$; [3]

(b) $(1+i)^{10}$; [3]

(c) $\left(\dfrac{1+i}{\sqrt{2}}\right)^{12}$. [3]

13. Given $z = 3\operatorname{cis}\frac{3\pi}{4}$, giving each answer in modulus–argument form with $-\pi < \theta \le \pi$:

(a) find z^2; [2]

(b) find z^3. [2]

14. By writing $-2i$ in modulus–argument form, solve $z^2 = -2i$, giving the roots in the form $a+bi$. [4]

15. Find the square roots of each number in the form $a+bi$.

(a) $3+4i$ [4]

(b) $-21-20i$ [4]

16. The equation $z^2+bz+c=0$, where $b, c \in \mathbb{R}$, has $\dfrac{5}{2-i}$ as a root. Find b and c. [5]

17. Find the three cube roots of $8i$, giving each

(a) in modulus–argument form; [3]

(b) in exact Cartesian form. [3]

18. Find all solutions of $z^3 = -4\sqrt{3}+4i$, giving each in modulus–argument form with $-\pi < \theta \le \pi$. [4]

19.

(a) Find the four fourth roots of -16, giving each in the form $a+bi$. [5]

(b) Find all solutions of $z^6 = 64$, giving each in the form $a+bi$, and show that they sum to zero. [4]

20. Let $z_1 = 4\operatorname{cis}\frac{5\pi}{6}$ and $z_2 = 2\operatorname{cis}\frac{\pi}{4}$.

(a) Find $z_1 z_2$ and $\dfrac{z_1}{z_2}$, giving each in modulus–argument form with $-\pi < \theta \le \pi$. [4]

(b) Describe fully the single transformation of the Argand diagram that maps the point representing z to the point representing zz_2. [2]

(c) The point A represents $w = 3+i$. Point B is the image of A under an anticlockwise rotation of $\frac{2\pi}{3}$ about the origin O. Find the complex number that B represents, in exact Cartesian form. [3]

(d) Find the area of triangle OAB. [2]

21. GDC Let $w = 3 + 4i$.

(a) Write w in the form $re^{i\theta}$, giving θ correct to 3 s.f. [2]
(b) Find the five solutions of $z^5 = w$ in the form $re^{i\theta}$, with $-\pi < \theta \leq \pi$, giving r and θ correct to 3 s.f. [4]
(c) Find the solution that lies in the third quadrant, in the form $a + bi$ with a and b correct to 3 s.f. [2]
(d) The five solutions are the vertices of a regular pentagon. Find its area. [2]

22. Given $z = \cos\theta + i\sin\theta$ with $\cos\theta \neq 0$, show that $\dfrac{z^2 - 1}{z^2 + 1} = i\tan\theta$. [4]

23. Prove De Moivre's theorem, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, for all $n \in \mathbb{Z}^+$ by mathematical induction. [7]

24. Use De Moivre's theorem to show that $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. [6]

25. Use De Moivre's theorem to show that $\tan 4\theta = \dfrac{4t - 4t^3}{1 - 6t^2 + t^4}$, where $t = \tan\theta$. [7]

26. One cube root of a complex number w is $1 + \sqrt{3}\,i$. Find w and the other two cube roots. [6]

27. In this question you will investigate the fifth roots of unity, the solutions of $z^5 = 1$.

(a) Write down the five roots in the form $\operatorname{cis}\theta$. [3]
(b) Show that the five roots sum to zero, using the sum of a geometric series. [3]
(c) Let $\omega = \operatorname{cis}\frac{2\pi}{5}$. Show that every fifth root of unity is a power of ω. [2]
(d) Deduce that $\omega + \omega^2 + \omega^3 + \omega^4 = -1$. [2]
(e) The five roots are plotted on an Argand diagram and joined to form a regular pentagon. Show that its area is $\frac{5}{2}\sin\frac{2\pi}{5}$. Given that $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$, find the exact area in the form $p\sqrt{10 + 2\sqrt{5}}$, where $p \in \mathbb{Q}$. [5]

28. Let $z = \operatorname{cis}\theta$.

(a) Show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ for any $n \in \mathbb{Z}^+$. [3]
(b) Expand $\left(z + \dfrac{1}{z}\right)^4$ using the binomial theorem, and hence show that $16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$. [5]
(c) Deduce that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. [2]
(d) Hence evaluate $\displaystyle\int_0^{\pi/2} \cos^4\theta \, d\theta$ exactly. [4]
(e) Explain what this method achieved that integrating $\cos^4\theta$ directly would not. [2]

29. The polynomial $P(z) = z^4 - 2z^3 + 14z^2 - 18z + 45$ has real coefficients, and $1 + 2i$ is a root.

(a) Write down another root of $P(z)$, and give a reason. [2]
(b) Find a real quadratic factor of $P(z)$. [3]
(c) Find the other quadratic factor. [3]
(d) Hence write down all four roots of $P(z)$. [3]
(e) The four roots are plotted on an Argand diagram. Describe the symmetry of the picture, and state the feature of $P(z)$ that guarantees it. [3]

Challenge Questions

30. Chords of a regular polygon. The nth roots of unity, $1, \omega, \omega^2, \dots, \omega^{n-1}$ with $\omega = \operatorname{cis} \frac{2\pi}{n}$, are the vertices of a regular n-sided polygon inscribed in the unit circle. From the vertex 1, draw the $n-1$ chords to the other vertices. The length of the chord to ω^k is $|1-\omega^k|$. This question asks for the product of those $n-1$ lengths, for every $n \geq 2$.

(a) Take $n=3$. Find $|1-\omega|$ and $|1-\omega^2|$ in exact form, and show that their product is 3. [3]
(b) Take $n=4$, so the vertices are $1, i, -1, -i$. Find the three chord lengths from 1 and their product. Using (a) and (b), conjecture the product of the $n-1$ chord lengths for a general n. [3]
(c) Explain why $z^n - 1 = (z-1)(z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$ for every complex z. By also factorising $z^n - 1 = (z-1)\left(1+z+z^2+\cdots+z^{n-1}\right)$, deduce that

$$(z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1}) = 1+z+z^2+\cdots+z^{n-1}$$

for every complex z, including $z=1$. [4]
(d) Hence prove that $\displaystyle\prod_{k=1}^{n-1}(1-\omega^k) = n$, and deduce your conjecture from (b). [3]
(e) Show that $1 - \operatorname{cis}\theta = -2i\sin\frac{\theta}{2}\operatorname{cis}\frac{\theta}{2}$, and deduce that $|1-\omega^k| = 2\sin\dfrac{k\pi}{n}$ for $k = 1, 2, \dots, n-1$. Hence show that

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \frac{n}{2^{n-1}},$$

and check this directly for $n=6$. [5]
(f) Now let $P = \operatorname{cis}\theta$ be any point on the circle. Investigate the product of the distances from P to all n vertices: find a formula in terms of θ, its greatest value, and where on the circle P must be to attain it. [3]

31. Adding waves. Two loudspeakers play the same pure note. At a listening point the two arrivals are $A\cos\omega t$ and $B\cos(\omega t+\phi)$, where $A>0$, $B>0$, and ϕ is the phase difference caused by the extra distance one sound has travelled. Adding them with trigonometry is lengthy; with complex numbers each wave becomes a vector in the Argand diagram, and the waves add as vectors do.

(a) Show that $A\cos\omega t + B\cos(\omega t+\phi) = \mathrm{Re}\Big[(A+B\,\mathrm{cis}\,\phi)\,\mathrm{cis}\,\omega t\Big]$. [3]

(b) Deduce that the combined sound is a wave of the same frequency, with amplitude $|A+B\,\mathrm{cis}\,\phi|$ and phase $\arg(A+B\,\mathrm{cis}\,\phi)$. [2]

(c) Two speakers each have amplitude 3 and the phase difference is $\frac{\pi}{3}$. Find the amplitude and phase of the combined sound, in exact form. [4]

(d) Now let both amplitudes equal A, and take $-\pi \le \phi \le \pi$. Show that $A + A\,\mathrm{cis}\,\phi = 2A\cos\dfrac{\phi}{2}\,\mathrm{cis}\,\dfrac{\phi}{2}$. Deduce that the combined amplitude is $2A\cos\dfrac{\phi}{2}$, that the phase is $\dfrac{\phi}{2}$ when $-\pi<\phi<\pi$, and state what happens when $\phi=\pm\pi$. [4]

(e) Use (d) to explain what a listener hears when $\phi=0$ and when $\phi=\pi$, and why noise-cancelling headphones work by generating a wave with $\phi=\pi$. [3]

(f) Now n speakers in a row each have amplitude A, and each arrival is a phase ϕ behind the one before, so the arrivals are $A\cos(\omega t + k\phi)$ for $k = 0, 1, \dots, n-1$, with $0<\phi<2\pi$. Use the sum of a geometric series in $\mathrm{cis}\,\phi$ to show that the combined amplitude is

$$A\left|\frac{\sin(n\phi/2)}{\sin(\phi/2)}\right|.$$

(Hint: $\mathrm{cis}\,\alpha - 1 = 2i\sin\frac{\alpha}{2}\,\mathrm{cis}\,\frac{\alpha}{2}$, found by taking out the factor $\mathrm{cis}\,\frac{\alpha}{2}$ as in (d)) [4]

(g) For $n=4$, find every ϕ in $0<\phi<2\pi$ that gives silence. Then investigate how the loud region around $\phi=0$ changes as n increases, and suggest why an array of many speakers can aim sound in one direction. [3]

32. From a circle to an aeroplane wing. The map $w = z + \dfrac{1}{z}$ is called the **Joukowski transformation**. Early aerodynamicists used it because it turns circles, whose airflow is easy to calculate, into wing-shaped curves, whose airflow is not.

(a) Let $z = \mathrm{cis}\,\theta$, so $|z|=1$. Show that $w = 2\cos\theta$, and hence that the unit circle maps onto the line segment from -2 to 2. [4]

(b) Now let $z = r\,\mathrm{cis}\,\theta$ with $r>1$. Writing $w = x+iy$, show that

$$x = \left(r+\tfrac{1}{r}\right)\cos\theta, \qquad y = \left(r-\tfrac{1}{r}\right)\sin\theta,$$

and deduce that $\dfrac{x^2}{\left(r+\frac{1}{r}\right)^2} + \dfrac{y^2}{\left(r-\frac{1}{r}\right)^2} = 1.$ [5]

(c) That equation describes an ellipse with semi-axes $r+\frac{1}{r}$ and $r-\frac{1}{r}$. Find both for $r=2$. [2]

(d) Describe what happens to the ellipse as $r\to1^+$, and reconcile your answer with part (a). [3]

(e) A wing needs a rounded front and a sharp trailing edge. Let $0<a<1$ and let Γ be the circle with centre $-a$ and radius $1+a$. Show that Γ passes through $z=1$, and that

$z = -1$ lies inside Γ. [2]

(f) The derivative $\dfrac{dw}{dz} = 1 - \dfrac{1}{z^2}$ is zero only at $z = \pm 1$, and at a point where it is zero the map doubles angles instead of preserving them. Use this and (e) to explain why the image of Γ has a sharp point at $w = 2$ and is smooth everywhere else. [3]

(g) Real wings are curved (cambered) as well as thick. Investigate the image of the circle through $z = 1$ whose centre is $-a + bi$, with $a > 0$ and $b > 0$: state its radius, and describe how a and b separately change the shape of the image. [2]

33. How a phone hears a chord. A microphone records a sound as a list of numbers, one per sampling instant. To find which frequencies are present, the recording is compared against the roots of unity. For four samples x_0, x_1, x_2, x_3, the **discrete Fourier transform** is

$$X_k = \sum_{n=0}^{3} x_n \, i^{-nk}, \qquad k = 0, 1, 2, 3,$$

where $i = \operatorname{cis} \frac{2\pi}{4}$ is the fourth root of unity next to 1. Each X_k measures how strongly the frequency k is present, meaning k complete cycles across the four samples.

(a) Write down the four fourth roots of unity and verify that they sum to zero. [2]

(b) A steady tone with no variation gives samples $(1, 1, 1, 1)$. Compute X_0, X_1, X_2, X_3. [4]

(c) Explain, using part (a), why every X_k with $k \neq 0$ had to vanish in (b) (for $k = 2$, note that $i^{-2} = -1$), and say what X_0 measures. [3]

(d) A signal that rises and falls once across the four samples gives $(1, 0, -1, 0)$. Compute X_0, X_1, X_2, X_3. [4]

(e) Interpret your answer to (d): which frequency is present, and what does $X_0 = 0$ tell you about the signal? [3]

(f) A chord is two notes at once. The samples $(2, -1, 0, -1)$ are the sum of the signal in (d) and the signal $(1, -1, 1, -1)$. Compute the transform of $(2, -1, 0, -1)$, and explain how it shows both notes. [3]

(g) Prove that the samples can be recovered from the transform by

$$x_n = \frac{1}{4} \sum_{k=0}^{3} X_k \, i^{nk}, \qquad n = 0, 1, 2, 3,$$

and check this for x_0 in (f). [4]

(h) Suggest how the transform and the formula in (g) extend to N samples. Then investigate what the transform gives for a tone that does not complete a whole number of cycles across the samples. [2]

34. The escape radius of the Mandelbrot set. Fix a complex number c, start at $z_0 = 0$, and iterate

$$z_{n+1} = z_n^2 + c.$$

For some c the sequence stays bounded forever; for others it grows without bound. The set of c for which it stays bounded is the **Mandelbrot set**, and a computer draws it by testing one c per pixel. This question finds the circle outside which no point of the set can lie, and the test a computer uses to stop.

(a) Take $c = i$. Compute z_1, z_2, z_3 and z_4, and show that the sequence then repeats with period 2. Is i in the set? [4]

(b) Take $c = 1$. Compute z_1, z_2, z_3, z_4 and say what is happening. [2]

(c) Show that if $|z| > 2$ and $|z| \geq |c|$ then $\left|z^2 + c\right| \geq |z|(|z| - 1) > |z|$. *(Hint: $|z^2 + c| \geq |z|^2 - |c|$)* [4]

(d) Suppose $|c| > 2$. Show that z_1 satisfies both conditions of (c), and prove by induction that $|z_n| \geq |c|(|c| - 1)^{n-1}$ for $n \geq 1$. Deduce that every c in the Mandelbrot set satisfies $|c| \leq 2$. [4]

(e) Now suppose $|c| \leq 2$, and that $|z_m| > 2$ for some m. Use (c) to show that the sequence then grows without bound. Combining this with (d), explain why a computer can stop testing a pixel, whatever c is, as soon as $|z_n| > 2$. [3]

(f) Investigate which real numbers c lie in the Mandelbrot set. Use the fixed points of $z \mapsto z^2 + c$ and a few iterations to suggest the interval, and justify at least one of its endpoints. [3]