CALCULUS

Chapter 16

Differential Equations

16

SYLLABUS 5.18 5.19 HL only

A cup of coffee cools fastest when it is hottest. Take it from the machine at 80°C and its temperature falls quickly at first; near room temperature it hardly changes. The *rate* at which it cools depends on how far above the room its temperature currently is.

That sentence is a differential equation. Written in symbols, the rate of cooling $\frac{d\theta}{dt}$ is proportional to the gap $\theta - 20$ between the coffee and the 20-degree room:

$$\frac{d\theta}{dt} = -k(\theta - 20).$$

Notice what kind of equation this is. The unknown is not a number. It is a *function*, $\theta(t)$, and the equation links that function to its own derivative. Many laws of nature have this form: they say how fast something grows, cools, decays or moves.

A **differential equation** relates a function to its derivatives. To **solve** one is to find the function itself, recovering $\theta(t)$ from a statement about $\frac{d\theta}{dt}$.

That is the point of solving one: it turns a rule about change into a prediction. A law of nature tells you what is happening at this instant; solving its differential equation tells you how the quantity will change from then on.

By the end of this chapter you will be able to solve such an equation exactly by three methods: separating the variables, substituting to make them separate, and using an integrating factor. You will also write a function as an infinite polynomial, called a Maclaurin series. That gives a fourth route to a solution, and a way to approximate a function. When no exact solution exists, you will solve the equation approximately by Euler's method. The coffee equation is solved in the first section.

> A calculator is assumed throughout this chapter. The EXACT badge marks the questions whose answer must be left exact.

16.1 Separable Variables

Differential equations appear wherever a law describes a rate of change, and that covers much of science. A few examples show the range.

- **Cooling.** $\dfrac{d\theta}{dt} = -k(\theta - \theta_{\text{room}})$: a hot drink, an engine, or a body found at a crime scene.
- **Radioactive decay.** $\dfrac{dN}{dt} = -\lambda N$: the basis of carbon dating.
- **Population growth.** $\dfrac{dn}{dt} = kn(M - n)$: growth that slows as the population fills the space available.
- **Medicine.** $\dfrac{dC}{dt} = -kC$: a drug leaving the bloodstream, which sets how often a dose must be repeated.
- **Falling through air.** $\dfrac{dv}{dt} = g - kv^2$: why a skydiver reaches a steady terminal speed.
- **Epidemics.** Three linked differential equations predict how an infection spreads through a population.

Each one says the same thing in different words: the rate is known, and the quantity is wanted.

Two words describe the equations in this chapter, and both are settled by looking at the equation.

- The **order** is the highest derivative that appears. Every equation here contains $\dfrac{dy}{dx}$ and no higher derivative, so all of them are **first order**.
- An equation is **linear** when y and its derivatives appear only to the first power, and are never multiplied by one another. So $\dfrac{dy}{dx} + xy = 1$ is linear, and $\dfrac{dy}{dx} = 2xy^2$ is not.

The three exact methods in this chapter are chosen by which form an equation takes, so these two points are checked first.

16.1.1 Separating the variables

The simplest case can be rearranged so that each variable sits on its own side. A first-order equation is **separable** if it can be written as a function of y times a function of x:

$$\frac{dy}{dx} = g(x)\,h(y).$$

Then you separate and integrate both sides. Treating $\dfrac{dy}{dx}$ as a ratio is informal but valid here: gather all the y terms with dy and all the x terms with dx.

KEY

$$\frac{dy}{dx} = g(x)\,h(y) \quad\implies\quad \int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

One unknown constant appears, giving the **general solution**, a family of curves. An **initial condition** fixes the constant and selects the **particular solution**. A particular solution holds only on the interval that contains the initial point and on which its formula is defined: if the formula is undefined at some value of x, the solution stops there.

When integration gives $\ln|y-a| = f(x) + C$, exponentiate both sides and write $y - a = Ae^{f(x)}$. The constant $A = \pm e^{C}$ replaces the modulus sign and the old constant together.

Worked through. To solve $\frac{dy}{dx} = 2xy^2$ given $y(0) = 1$, separate the variables and integrate:

$$\int \frac{1}{y^2}\,dy = \int 2x\,dx \implies -\frac{1}{y} = x^2 + C.$$

Apply $y(0) = 1$: $-1 = 0 + C \implies C = -1$. So $-\dfrac{1}{y} = x^2 - 1$, giving

$$y = \frac{1}{1-x^2}, \qquad -1 < x < 1.$$

The solution is valid on $-1 < x < 1$, the interval containing the starting point $x = 0$ on which $1 - x^2 \neq 0$. Apply the initial condition as early as possible; carrying C through more steps than necessary makes errors more likely.

EXAMPLE 16.1

A cup of coffee obeys $\dfrac{d\theta}{dt} = -k(\theta - 20)$ with $\theta(0) = 80$. Solve for $\theta(t)$.
Separate and integrate:

$$\int \frac{1}{\theta - 20}\,d\theta = \int -k\,dt \implies \ln|\theta - 20| = -kt + C.$$

Exponentiating, $\theta - 20 = Ae^{-kt}$. The condition $\theta(0) = 80$ gives $A = 60$, so

$$\theta(t) = 20 + 60e^{-kt}.$$

The temperature decays exponentially toward the room's 20 degrees, never quite reaching it. This is the behaviour described at the start of the chapter.

Separation also solves a standard population model in ecology. In the 1960s, once the cattle disease rinderpest was eliminated around the Serengeti, the wildebeest herd began to grow, and its growth obeyed the **logistic equation**: growth proportional both to the herd itself and to the room left below the **carrying capacity**, the largest population the land can support.

EXAMPLE 16.2

The Serengeti wildebeest population n (in millions) satisfies $\dfrac{dn}{dt} = kn(1.4 - n)$, with $n(0) = 0.25$ in 1961. Solve for $n(t)$.

Separate, and split the left side with the partial fractions of Ch. 6:

$$\int \frac{dn}{n(1.4-n)} = \int k\,dt, \qquad \frac{1}{n(1.4-n)} = \frac{1}{1.4}\left(\frac{1}{n} + \frac{1}{1.4-n}\right).$$

Integrating,

$$\frac{1}{1.4}\ln\left|\frac{n}{1.4-n}\right| = kt + C \implies \frac{n}{1.4-n} = Be^{1.4kt}.$$

The initial condition gives $B = \dfrac{0.25}{1.15} \approx 0.217$, and solving for n:

$$n(t) = \frac{1.4}{1 + 4.6\,e^{-1.4kt}}.$$

This is the logistic S-curve: growth that is nearly exponential at first, then approaches the carrying capacity of 1.4 million. The bracket $(1.4 - n)$ is what slows it, since it decreases towards zero as n approaches 1.4. The real herd behaved this way, climbing to about 1.4 million by 1977 and fluctuating around that level since.

YOUR TURN 16.1 Separable Variables

Recognising the type of equation

1. Each equation is first order. State whether it is linear and whether it is separable, giving a reason for each.

(a) $\dfrac{dy}{dx} = x^2 y$

(b) $\dfrac{dy}{dx} = y^2 + x$

(c) $\dfrac{dy}{dx} + y\cos x = \sin x$

(d) $y\dfrac{dy}{dx} = x$

General solutions

2. Find the general solution of each equation.

(a) $\dfrac{dy}{dx} = \dfrac{x}{y}$

(b) $\dfrac{dy}{dx} = y$

(c) $\dfrac{dy}{dx} = \dfrac{1+y}{1+x}$

(d) $\dfrac{dy}{dx} = e^{x-y}$

Particular solutions and where they are valid

3. Find the particular solution of each equation.

(a) $\dfrac{dy}{dx} = xy$ with $y(0) = 2$.

(b) $\dfrac{dy}{dx} = 1 + y^2$ with $y(0) = 1$. State the interval containing $x = 0$ on which the solution is valid.

(c) $\dfrac{dy}{dx} = y \tan x$ with $y(0) = 4$, for $0 \le x < \frac{\pi}{2}$.

(d) $\dfrac{dy}{dx} = y^2 \cos x$ with $y(0) = 1$. State the interval containing $x = 0$ on which the solution is valid.

(e) Explain why a general solution contains one arbitrary constant, and what the initial condition does to it.

Models

4. A metal bar is taken from a furnace at $300°$C into a room at $25°$C. Its temperature $\theta\,°$C after t minutes satisfies $\dfrac{d\theta}{dt} = -0.05(\theta - 25)$.

(a) Solve the equation to find θ in terms of t.

(b) Find the time the bar takes to cool to $100°$C.

5. The number N of bacteria in a culture grows at a rate proportional to N. Initially $N = 500$, and after 2 hours $N = 800$.

(a) Write down a differential equation for N, and solve it to find N in terms of t, the time in hours.

(b) Find N after 5 hours.

6. A population of n thousand animals satisfies $\dfrac{dn}{dt} = 0.3n(2 - n)$, where t is the time in years, and $n(0) = 0.5$.

(a) Use partial fractions to show that $n = \dfrac{2}{1 + 3e^{-0.6t}}$.

(b) Find the time at which $n = 1.5$, and state the value that n approaches as t increases.

Concept check

7. A student solves $\dfrac{dy}{dx} = xy$, for $y > 0$, and writes

$$\ln y = \tfrac{1}{2}x^2 + C \quad \Rightarrow \quad y = e^{x^2/2} + C.$$

Identify the error, and give the correct general solution.

16.2 Homogeneous Differential Equations

Some equations cannot be separated as they stand, but a substitution makes them separable.

A first-order equation is **homogeneous** if the right-hand side depends only on the ratio $\dfrac{y}{x}$, and not on x and y separately.

Equations of this kind appear when a problem has no fixed size, only directions and proportions.

- **Pursuit curves.** A ferry always aims at the dock while a current pushes it sideways. At each moment only the *direction* from the ferry to the dock matters, and that direction is set by $\dfrac{y}{x}$ alone. You will solve the ferry in the challenge problems.
- **Reflector shapes.** The curve of a car headlight or a satellite dish is fixed by requiring every ray from the focus to leave parallel. That condition involves only angles, so the equation depends on the ratio.
- **Scale-free rules.** If multiplying both quantities by the same factor leaves a rule unchanged, the rule can only involve their ratio, and the equation is homogeneous.

The substitution $y = vx$ turns such an equation into a separable one. Since $y = vx$, the product rule gives $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

KEY For $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$, substitute $y = vx$, so $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

After the substitution the equation becomes $v + x\dfrac{dv}{dx} = f(v)$, that is $x\dfrac{dv}{dx} = f(v) - v$. This has v on one side and x on the other, so it is always separable. The method has four steps.

1. Check that the right side depends only on $\dfrac{y}{x}$: divide its top and bottom by the highest power of x.
2. Substitute $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.
3. Separate the variables and integrate.

4. Replace v by $\dfrac{y}{x}$.

Worked through. To solve $\frac{dy}{dx} = \frac{x+y}{x}$, rewrite the right side in terms of $\frac{y}{x}$: $\dfrac{x+y}{x} = 1 + \dfrac{y}{x}$. Substitute $y = vx$, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$:

$$v + x\frac{dv}{dx} = 1 + v \implies x\frac{dv}{dx} = 1.$$

This is separable:

$$\int dv = \int \frac{1}{x}\,dx \implies v = \ln|x| + C.$$

Finally replace $v = \dfrac{y}{x}$:

$$y = x(\ln|x| + C).$$

Here the v on each side cancelled. In general it does not, but $x\dfrac{dv}{dx} = f(v) - v$ is still separable.

EXAMPLE 16.3
Solve $\dfrac{dy}{dx} = \dfrac{2x^2 + y^2}{xy}$ for $x > 0$.
Divide the top and bottom by x^2 to show that the right side depends only on the ratio:

$$\frac{2x^2 + y^2}{xy} = \frac{2 + (y/x)^2}{y/x}.$$

Substitute $y = vx$, so $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$:

$$v + x\frac{dv}{dx} = \frac{2 + v^2}{v} \implies x\frac{dv}{dx} = \frac{2 + v^2}{v} - v = \frac{2}{v}.$$

Again the v terms cancel, leaving a separable equation:

$$\int v\,dv = \int \frac{2}{x}\,dx \implies \frac{v^2}{2} = 2\ln x + C.$$

Replace $v = \dfrac{y}{x}$ and clear the fraction:

$$y^2 = 2x^2(2\ln x + C).$$

YOUR TURN 16.2 Homogeneous Differential Equations

Recognising a homogeneous equation

8. State whether each equation is homogeneous, giving a reason.

(a) $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$

(b) $\dfrac{dy}{dx} = \dfrac{x^2+y}{x}$

(c) $\dfrac{dy}{dx} = \dfrac{y^2+xy}{x^2}$

General solutions

9. Use the substitution $y = vx$ to solve each equation, for $x > 0$.

(a) $\dfrac{dy}{dx} = \dfrac{y}{x}$

(b) $\dfrac{dy}{dx} = \dfrac{y-x}{x}$

(c) $\dfrac{dy}{dx} = \dfrac{x-y}{x}$

(d) $\dfrac{dy}{dx} = \dfrac{x+2y}{x}$

10.

(a) Show that $\dfrac{dy}{dx} = \dfrac{x^2+3y^2}{2xy}$ is homogeneous, by writing the right-hand side in terms of $\dfrac{y}{x}$.

(b) Hence find its general solution, for $x > 0$.

Particular solutions

11. Find the particular solution of each equation, for $x > 0$.

(a) $\dfrac{dy}{dx} = \dfrac{2y-x}{x}$ with $y(1) = 3$.

(b) $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{xy}$ with $y(1) = 2$.

Concept check

12. A student solves $\dfrac{dy}{dx} = \dfrac{x+y}{x}$, for $x > 0$, with the substitution $y = vx$, and writes $\dfrac{dy}{dx} = \dfrac{dv}{dx}$, so that $\dfrac{dv}{dx} = 1 + v$. Identify the error, and find the correct general solution.

16.3 The Integrating Factor

A linear equation often appears when a quantity is added and removed at the same time, and each rate depends at most linearly on the amount present.

- **Mixing tanks.** Liquid flows into a tank and drains out of it, and the dissolved substance obeys a linear equation. When the volume itself changes, the equation is linear but not separable, so the integrating factor is the method that applies.
- **Drug infusion.** A drug enters the blood at a steady rate through a drip, while the body clears it at a rate proportional to the amount already present.
- **Electrical circuits.** In a circuit with a resistor and a capacitor, the charge satisfies a linear equation driven by the applied voltage.
- **Savings with deposits.** An account earns interest continuously while money is paid in at a steady rate.

A **linear** first-order differential equation has the form

$$\frac{dy}{dx} + P(x)\,y = Q(x).$$

It is usually not separable. The method is to multiply through by a chosen function, the **integrating factor**, which turns the left side into the derivative of a single product. The coefficient of $\frac{dy}{dx}$ must be 1 first: if the equation starts as $x\frac{dy}{dx} + 2y = x^3$, divide by x before reading off $P(x)$.

KEY · IB For $\frac{dy}{dx} + P(x)y = Q(x)$, the integrating factor is

$$I = e^{\int P(x)\,dx}.$$

After multiplying by I, the left side becomes exactly $\frac{d}{dx}(Iy)$. The reason is the product rule: since $I = e^{\int P\,dx}$, its derivative is $I' = PI$, so

$$\frac{d}{dx}(Iy) = I\frac{dy}{dx} + I'y = I\frac{dy}{dx} + PIy = I\left(\frac{dy}{dx} + Py\right).$$

You then integrate both sides directly.

Worked through. In $\frac{dy}{dx} + \frac{1}{x}y = x^2$, for $x > 0$, $P(x) = \frac{1}{x}$, so the integrating factor is

$$I = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x.$$

Multiply through by x:

$$x\frac{dy}{dx} + y = x^3 \implies \frac{d}{dx}(xy) = x^3.$$

The left side is now a single derivative. Integrate both sides:

$$xy = \frac{x^4}{4} + C \implies y = \frac{x^3}{4} + \frac{C}{x}.$$

The method exists to create that $\frac{d}{dx}(xy)$ on the left; everything after is ordinary integration.

EXAMPLE 16.4

Solve $\dfrac{dy}{dx} + y = e^x$.

$P(x) = 1$, so $I = e^{\int 1\,dx} = e^x$. Multiply through:

$$\frac{d}{dx}(e^x y) = e^x \cdot e^x = e^{2x}.$$

Integrate: $e^x y = \frac{1}{2}e^{2x} + C$, hence $y = \frac{1}{2}e^x + Ce^{-x}$.

The last worked example of this section is a mixing problem of the first kind. To model a well-stirred mixing tank, write $\dfrac{dS}{dt} = (\text{rate in}) - (\text{rate out})$, where S is the amount of dissolved substance. It enters at the inflow concentration times the inflow rate, and leaves at the current concentration $\dfrac{S}{V}$ times the outflow rate. When the two flow rates differ, the volume changes too: $V(t) = V_0 + (\text{inflow rate} - \text{outflow rate})\,t$.

EXAMPLE 16.5

A 100-litre vat in a brewery holds 5 kg of dissolved sugar. Syrup carrying 0.2 kg of sugar per litre flows in at 1 litre per minute, the vat is kept well stirred, and the mixture drains at 2 litres per minute. Find the sugar $S(t)$ in the vat at time t minutes.

The volume shrinks: $V(t) = 100 - t$. Sugar arrives at $0.2 \times 1 = 0.2$ kg min^{-1} and leaves at concentration times outflow, $\dfrac{S}{100-t} \times 2$. So

$$\frac{dS}{dt} = 0.2 - \frac{2S}{100-t} \implies \frac{dS}{dt} + \frac{2}{100-t}S = 0.2.$$

The integrating factor is

$$I = e^{\int \frac{2}{100-t}\,dt} = e^{-2\ln(100-t)} = (100-t)^{-2}.$$

Multiplying through turns the left side into a single derivative:

$$\frac{d}{dt}\Big(S\,(100-t)^{-2}\Big) = 0.2\,(100-t)^{-2} \implies S\,(100-t)^{-2} = \frac{0.2}{100-t} + C.$$

So $S = 0.2(100-t) + C(100-t)^2$, and $S(0) = 5$ gives $C = -0.0015$:

$$S(t) = 0.2(100-t) - 0.0015(100-t)^2.$$

Check the endpoints: $S(0) = 5$, and $S(100) = 0$, an empty vat holds no sugar. In between, the sugar first *rises*, to about 6.7 kg near $t = 33$, and then falls. That rise is not obvious from the description, and it is the equation rather than intuition that reveals it.

16.3.1 Choosing a method

Three exact methods are now available, and a question may not say which to use. The form of the equation decides it, and the test is quick.

What the equation looks like	Method	Example
The variables can be split, $\frac{dy}{dx} = g(x)h(y)$	separate and integrate	$\frac{dy}{dx} = 2xy^2$
The right side depends only on $\frac{y}{x}$	substitute $y = vx$	$\frac{dy}{dx} = \frac{x+y}{x}$
Linear in y: $\frac{dy}{dx} + P(x)y = Q(x)$	integrating factor $e^{\int P\,dx}$	$\frac{dy}{dx} + \frac{1}{x}y = x^2$
None of the above	Euler's method, numerically	$\frac{dy}{dx} = x^2 + y^2$

Try them in that order, because each test is quicker than the one after it. Note that the first two overlap: an equation can be both separable and homogeneous, in which case separating directly is the shorter route. A linear equation can overlap too. When Q/P is constant, $\frac{dy}{dx} + P(x)y = Q(x)$ rearranges to $\frac{dy}{dx} = P(x)\left(\frac{Q}{P} - y\right)$ and separates at once, which is how $\frac{dy}{dx} + y = 4$ and the cooling equation of §16.1 are solved. In general, though, Q/P is not constant, the equation does not separate, and the integrating factor is the method to use.

YOUR TURN 16.3 The Integrating Factor

Solving a linear equation

13. Solve each linear equation.

(a) $\dfrac{dy}{dx} + y = 4$

(b) $\dfrac{dy}{dx} + 2y = 6$

(c) $\dfrac{dy}{dx} + 2y = 4x$

(d) $\dfrac{dy}{dx} - y = e^{2x}$

14.

(a) Write down the integrating factor for $\dfrac{dy}{dx} + \dfrac{3}{x}y = x^2$, for $x > 0$.

(b) Solve $\dfrac{dy}{dx} + \dfrac{2}{x}y = x$ for $x > 0$.

(c) Solve $\dfrac{dy}{dx} - \dfrac{1}{x}y = x^2$ with $y(1) = 1$, for $x > 0$.

(d) Explain what the integrating factor is chosen to achieve on the left-hand side.

Rearranging into standard form first

15. Divide by the coefficient of $\dfrac{dy}{dx}$ first, then solve each equation.

(a) $x\dfrac{dy}{dx} + 3y = x^2$, for $x > 0$.

(b) $(x^2+1)\dfrac{dy}{dx} + 2xy = 4x$

(c) $\dfrac{dy}{dx} + y\tan x = \cos x$, for $0 \le x < \frac{\pi}{2}$.

(d) $\cos x\dfrac{dy}{dx} + y\sin x = 1$, for $0 \le x < \frac{\pi}{2}$.

A mixing tank

16. A tank holds 50 litres of water in which 2 kg of salt is dissolved. Brine containing 0.1 kg of salt per litre flows in at 3 litres per minute, the tank is kept well stirred, and the mixture drains at 2 litres per minute. Let S kg be the salt in the tank after t minutes.

(a) Show that $\dfrac{dS}{dt} + \dfrac{2S}{50+t} = 0.3$.

(b) Solve the equation to find S in terms of t.

(c) Find the salt in the tank after 10 minutes.

Choosing a method

17. For each equation, state which method applies, and find the general solution.

(a) $\dfrac{dy}{dx} = xe^{-y}$

(b) $\dfrac{dy}{dx} - \dfrac{2y}{x} = x^3$, for $x > 0$.

(c) $\dfrac{dy}{dx} = \dfrac{y^2 + 2xy}{x^2}$, for $x > 0$.

Concept check

18. A student solves $\dfrac{dy}{dx} + \dfrac{y}{x} = 1$, for $x > 0$, with the integrating factor x, and writes

$$xy = \tfrac{1}{2}x^2 + C \quad \Rightarrow \quad y = \tfrac{1}{2}x + C.$$

Identify the error, and show that the student's answer does not satisfy the equation unless $C = 0$.

16.4 Maclaurin Series

Many functions can be written as an infinite polynomial. This idea stands apart from the three methods above, and it has two uses: it approximates a function, and it gives a fourth route to solving a differential equation.

There is a reason a polynomial is wanted at all. A polynomial is evaluated using only addition and multiplication, which are fast and exact to program. A function such as $\sin x$ has no such direct recipe, so computer software often evaluates it from a polynomial approximation built in this way. (Many calculators use a different method based on repeated addition and halving, but the aim is the same: reduce the function to arithmetic.)

The method is to match the function at a single point. Near $x = 0$, we look for a polynomial with the same value as the function, then the same gradient, then the same concavity, then the same higher derivatives. The more derivatives agree, the wider the interval around 0 on which the polynomial stays close to the curve. The result is the **Maclaurin series**.

The coefficients are completely determined by the function. Suppose the series exists and write it as

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Putting $x = 0$ makes every term except the first equal to zero, so $a_0 = f(0)$. Differentiating gives $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$, and $x = 0$ now leaves $a_1 = f'(0)$. Differentiating again gives $f''(0) = 2a_2$, and once more $f'''(0) = 6a_3 = 3!\, a_3$. Each differentiation removes the constant term and multiplies each remaining coefficient by its power, so in general

$f^{(n)}(0) = n!\, a_n$.

KEY · IB

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Worked through. To find the Maclaurin series for $f(x) = e^x$, note that every derivative of e^x is e^x, and $e^0 = 1$, so $f^{(n)}(0) = 1$ for all n:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

This series can be taken as the definition of e^x, and it computes the function to any accuracy you want. It also explains why e^x is its own derivative: differentiate the series term by term and you get the same series back.

The series below are provided in the formula booklet. Each one equals its function exactly on an interval around 0: for every x in the case of e^x, $\sin x$ and $\cos x$, but only for $-1 < x \leq 1$ for $\ln(1+x)$ and for $|x| \leq 1$ for $\arctan x$. Truncating a series gives a polynomial approximation that is very accurate near $x = 0$.

KEY · IB

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

One more expansion is named by the syllabus but printed elsewhere in the booklet, so it is easy to overlook. The binomial family $(1+x)^p$ is a Maclaurin series like any other, and it is the extended binomial theorem of Ch. 2 (booklet 1.10) seen from this side.

KEY For $p \in \mathbb{Q}$ and $|x| < 1$,

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots.$$

Like $\ln(1+x)$ and $\arctan x$, it carries a validity condition, because, unless p is a non-negative integer, it does not converge for every x.

EXAMPLE 16.6

Use a Maclaurin series to approximate $\sqrt{1.1}$ to four decimal places.

Write $\sqrt{1.1} = (1+x)^{1/2}$ with $x = 0.1$, which satisfies $|x| < 1$. With $p = \frac{1}{2}$,

$$(1+x)^{1/2} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \cdots.$$

Substituting $x = 0.1$ gives

$$1 + 0.05 - 0.00125 + 0.0000625 = 1.0488125.$$

The true value is $\sqrt{1.1} = 1.0488088\ldots$, so four terms agree to four decimal places, 1.0488. Each term is smaller than the last by roughly a factor of x, which is why these series are useful: the smaller x is, the fewer terms you need.

The figure below shows the first three odd-degree truncations of the sine series: the line $y = x$, the cubic, and the quintic. Each new term extends the interval over which the polynomial and the curve agree.

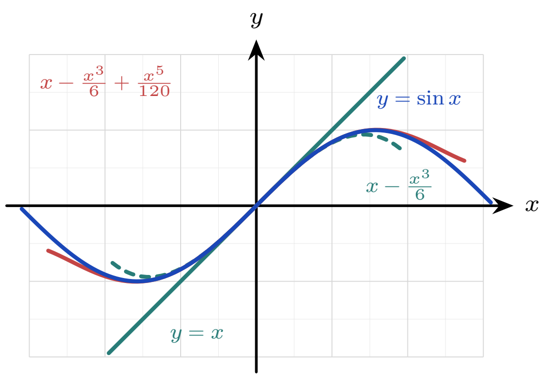

Figure 16.1 The first three odd-degree truncations of the sine series against $y = \sin x$. Each extra term widens the interval on which the polynomial and the curve agree, but every truncation eventually leaves the curve.

The same convergence can be watched numerically. Below are the successive truncations of the sine series at $x = 1$, against the true value $\sin 1 = 0.841471$.

terms up to	x	x^3	x^5	x^7
estimate	1	0.833333	0.841667	0.841468
error	0.159	0.0081	0.0002	0.0000027

Each new term divides the error by a factor of tens, and that factor grows as more terms are taken: the ratios along this row are about 20, 40 and 70. So a short polynomial is easy to compute, and each extra term gives more accuracy for a little more work.

16.4.1 Building new series

New series are usually not built from the definition. They are made from the standard five by four operations, all of which the syllabus names.

- **Substitute.** Replacing x with $-x^2$ in the series for e^x gives

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

- **Differentiate.** Differentiating the sine series term by term produces exactly the cosine series: the identity $\frac{d}{dx}\sin x = \cos x$ in polynomial form.
- **Integrate.** The geometric series of Ch. 2 gives $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots$ for $|x| < 1$, and integrating term by term yields $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$. The booklet's arctangent series is simply the geometric series integrated.
- **Multiply.** For a product, expand each factor and collect powers up to the degree you need.

Worked through. To find the Maclaurin series of $e^x \sin x$ up to the term in x^3, multiply the two standard series, keeping only terms of degree ≤ 3:

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(x - \frac{x^3}{6}\right) = x + x^2 + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \cdots = x + x^2 + \frac{x^3}{3} + \cdots$$

The work is careful collection of terms. Decide the target degree first, and discard anything above it before multiplying rather than after.

16.4.2 Small angles, limits, and series from differential equations

Truncating at the first term or two gives the **small-angle approximations** used throughout physics. For θ near 0, in radians,

$$\sin\theta \approx \theta, \qquad \cos\theta \approx 1 - \frac{\theta^2}{2}, \qquad \tan\theta \approx \theta.$$

These are not conveniences invented by physicists; they are the first one or two terms of the series, with the rest discarded. The pendulum formula depends on the first of them. For small swings the restoring force involves $\sin\theta$, and replacing $\sin\theta$ by θ is exactly what turns the equation of motion into simple harmonic motion, which can then be solved.

The series also measures the error of the approximation. The first term discarded is $\frac{\theta^3}{6}$, so that is roughly the error, and the relative error stays under 1% for swings up to about 14°. Beyond that the pendulum's real period moves away from the textbook formula, by an amount large enough to measure.

The same truncation resolves indeterminate limits: replace each function by the first terms of its series, and the limit can be read off.

EXAMPLE 16.7

Use Maclaurin series to evaluate $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

Expand the numerator with the series for e^x:

$$e^x - 1 - x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) - 1 - x = \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

Divide by x^2:

$$\frac{e^x - 1 - x}{x^2} = \frac{1}{2} + \frac{x}{6} + \cdots \xrightarrow{x\to 0} \frac{1}{2}.$$

The series resolves an indeterminate limit as neatly as L'Hôpital's rule, and it shows *why* the answer is $\frac{1}{2}$: that number is the leading coefficient.

A Maclaurin series can also be built directly *from a differential equation*, by differentiating the equation itself to obtain the derivatives at 0. This joins the two halves of the chapter.

Worked through. To find the Maclaurin series, up to the term in x^3, of the function that satisfies $\dfrac{dy}{dx} = 1 + y^2$ with $y(0) = 0$, take the derivatives at 0 from the equation itself. First, $y'(0) = 1 + 0 = 1$. Differentiating the equation,

$$y'' = 2y\,y' \implies y''(0) = 0, \qquad y''' = 2(y')^2 + 2y\,y'' \implies y'''(0) = 2.$$

Substitute these into the Maclaurin formula:

$$y = 0 + x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \cdots = x + \frac{x^3}{3} + \cdots$$

You may recognise the answer: $y = \tan x$ solves this equation, since $\frac{d}{dx}\tan x = \sec^2 x = 1 + \tan^2 x$, and these are indeed the first terms of its series. The equation was never solved. Each derivative at 0 was read off from the equation itself, one at a time, and that was enough to build the series.

TIP Maclaurin series suggest several good IA topics. Two worth exploring: Madhava's series $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$, which comes from setting $x = 1$ in the arctangent series, and why it converges so slowly that a better choice such as $\arctan\frac{1}{\sqrt{3}}$ reaches the same accuracy in a fraction of the terms; or how software evaluates $\sin x$ quickly, by first reducing the angle to a small interval and then using a short polynomial.

YOUR TURN 16.4 Maclaurin Series

Using the standard series

19. Find the first three non-zero terms of the Maclaurin series for each function.

(a) e^{2x}

(b) $\cos 3x$

(c) $\ln(1+2x)$

(d) e^{-x^2}

20. Find the first two non-zero terms of the Maclaurin series for each function.

(a) $\sin 2x$

(b) $\sin(x^2)$

(c) $\arctan 2x$

(d) $\sqrt{1+x}$

Series from the definition

21. Use the Maclaurin formula $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$, differentiating each function yourself, to find its series up to the term in x^4.

(a) $f(x) = \cos 2x$

(b) $f(x) = \ln(\cos x)$

Products of series

22. Find the Maclaurin series of each product up to the term in x^3.

(a) $e^x \cos x$

(b) $e^x \ln(1+x)$

Differentiating and integrating a series

23.

(a) Differentiate the series $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ term by term, and verify that the result is the expansion of $\frac{1}{1+x^2}$.

(b) Differentiate the series for $\ln(1+x)$ term by term, and verify that the result is the expansion of $\frac{1}{1+x}$.

(c) Integrate the geometric series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ term by term to find the first four terms of the series for $-\ln(1-x)$, for $|x| < 1$.

(d) Use the first three non-zero terms of the series for e^{-x^2} to estimate $\int_0^{0.5} e^{-x^2}\,dx$, correct to four decimal places.

Approximating values

24.

(a) Use the first three terms of the series for e^x to estimate $e^{0.1}$.

(b) Use the first three terms of the cosine series to estimate $\cos 0.2$ to four decimal places.

(c) Compare each estimate with the true value, and state which is the more accurate. Explain why.

25. The small-angle approximations are $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$ and $\tan\theta \approx \theta$, for θ in radians close to 0.

(a) Find the value that $\dfrac{\sin 3\theta \tan\theta}{1-\cos 2\theta}$ is close to when θ is small.

(b) Show that $\cos\theta + \theta\sin\theta \approx 1 + \dfrac{\theta^2}{2}$ when θ is small.

Limits by series

26. Use Maclaurin series to evaluate each limit.

(a) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$

(b) $\displaystyle\lim_{x\to 0} \frac{\ln(1+x) - x}{x^2}$

(c) $\displaystyle\lim_{x\to 0} \frac{x - \arctan x}{x^3}$

Series from a differential equation

27. A function satisfies $\dfrac{dy}{dx} = x + y$ with $y(0) = 1$.

(a) Write down $y'(0)$.

(b) By differentiating the equation, find $y''(0)$ and $y'''(0)$.

(c) Hence find the Maclaurin series for y up to the term in x^3.

(d) The exact solution is $y = 2e^x - x - 1$. Use the series to estimate $y(0.3)$, and compare it with the exact value.

28. A function satisfies $\dfrac{dy}{dx} = x - y^2$ with $y(0) = 1$.

(a) Find $y'(0)$, $y''(0)$ and $y'''(0)$.

(b) Hence find the Maclaurin series for y up to the term in x^3.

Concept check

29. A student replaces x with $3x$ in the cosine series and writes $\cos 3x = 1 - \frac{3x^2}{2} + \frac{3x^4}{24} - \cdots$. Identify the error, and give the correct first three terms.

30. A student estimates $\ln 3$ by putting $x = 2$ into $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$, and gets $2 - 2 + \frac{8}{3} \approx 2.67$. Explain why this estimate is not valid, and why taking more terms does not help.

16.5 Euler's Method

Not every differential equation can be solved exactly, and many of the equations that describe the real world cannot. When no exact solution exists, the solution can still be approximated numerically. **Euler's method** does this by moving along the curve in short straight steps, taking the gradient of each step from the differential equation itself.

The idea is the tangent line of Ch. 13. Near a known point the curve is close to its tangent, so moving a small distance h along the tangent lands close to the curve:

$$y \approx y_0 + \underbrace{f(x_0, y_0)}_{\text{gradient}} \times \underbrace{h}_{\text{step}} .$$

The differential equation supplies the gradient at whatever point you have reached, so the same move can be repeated from the new point, and again from the one after that.

KEY · IB

$$y_{n+1} = y_n + h\, f(x_n, y_n), \qquad x_{n+1} = x_n + h$$

Here h is the fixed **step length**. Each step has the correct gradient only at its starting point and moves away from the curve after that, so shorter steps stay closer, at the cost of needing more of them.

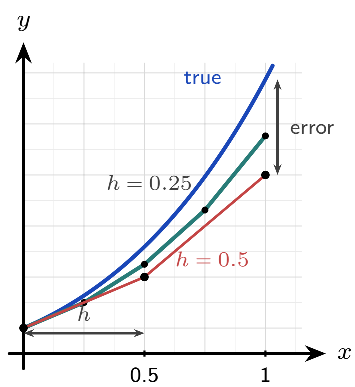

Figure 16.2 Euler's method on $\frac{dy}{dx} = x + y$ with $y(0) = 1$. Each straight step leaves the curve, which bends upward away from it, so the estimate falls short and the gap at $x = 1$ is the error. Halving the step length cuts that gap substantially, here from about 0.94 to 0.55.

In the figure, the gap at the far end is the *error*. The teal path takes steps half as long, and it ends closer to the curve.

The direction of the error follows from the concavity. Where the solution is concave up, each tangent step lies below the curve; where it is concave down, each step lies above it. For a small step length this is the usual pattern: Euler's method under-estimates a concave-up solution and over-estimates a concave-down one, so the sign of $\frac{d^2y}{dx^2}$ tells you which way your answer is wrong before you check it.

Worked through. Take the equation in the figure, $\frac{dy}{dx} = x + y$ with $y(0) = 1$, and estimate $y(0.3)$ with $h = 0.1$. Apply $y_{n+1} = y_n + h(x_n + y_n)$ three times. A table is the clearest way to organise the work, and it shows each step clearly to whoever marks the work.

n	x_n	y_n	$f = x_n + y_n$	$h\,f$
0	0	1	1	0.1
1	0.1	1.1	1.2	0.12
2	0.2	1.22	1.42	0.142
3	0.3	1.362		

Each y_n is the previous one plus the $h\,f$ on the line above. So $y(0.3) \approx 1.362$, against an exact value of about 1.3997.

The estimate is too small, and the reason is visible in the figure: this curve is concave up, so a tangent step lies below it and the estimate falls short, as the concavity pattern above predicts.

The method matters most when no exact solution is available. Make the resistance on a falling body depend on the shape of the object and on the air density, which changes with height, and the equation has no solution among the functions of this course, but Euler's method still produces a table of values. This is, in miniature, how weather forecasts are computed: equations too hard to solve exactly, stepped forward numerically, millions of times.

A quantity stays at a steady value only where its rate of change is zero. For an equation $\dfrac{dv}{dt} = f(v)$, solve $f(v) = 0$ to find the terminal or **equilibrium** value; this does not require solving the differential equation.

EXAMPLE 16.8

A skydiver falls from rest, and air resistance gives $\dfrac{dv}{dt} = 9.8 - 0.1v^2$ (in m s^{-1} and seconds). Use

Euler's method with $h = 0.5$ to estimate the speed after 1.5 seconds, and compare with the terminal velocity.

Apply $v_{n+1} = v_n + 0.5(9.8 - 0.1v_n^2)$ three times from $v_0 = 0$:

$$v_1 = 0 + 0.5(9.8) = 4.9$$
$$v_2 = 4.9 + 0.5(9.8 - 0.1(4.9)^2) = 4.9 + 0.5(7.399) = 8.60$$
$$v_3 = 8.60 + 0.5(9.8 - 0.1(8.60)^2) = 8.60 + 0.5(2.404) = 9.80$$

So $v(1.5) \approx 9.80$ m s^{-1}. The terminal velocity is where the acceleration is zero: $9.8 - 0.1v^2 = 0 \implies v = \sqrt{98} \approx 9.90$ m s^{-1}. The exact solution gives $v(1.5) \approx 8.93$ m s^{-1}, so this estimate is about 10% too high. The reason is the concavity pattern stated earlier in this section: the graph of v against t is concave down, so each tangent step lies above the curve and Euler's method overestimates. A smaller step length reduces the error. Each bracket $9.8 - 0.1v_n^2$ is the net force per unit mass, and it decreases towards zero as the speed increases.

YOUR TURN 16.5 Euler's Method

Carrying out Euler's method

31. Use Euler's method for each of the following.

(a) $\dfrac{dy}{dx} = 2x$, $y(0) = 1$; use $h = 0.5$ to estimate $y(1)$.

(b) $\dfrac{dy}{dx} = y$, $y(0) = 1$; use $h = 0.1$ to estimate $y(0.2)$.

(c) $\dfrac{dy}{dx} = x - y$, $y(0) = 1$; use $h = 0.2$ to estimate $y(0.4)$.

(d) $\dfrac{dy}{dx} = xy$, $y(1) = 1$; use $h = 0.1$ to estimate $y(1.2)$.

32.

(a) For $\dfrac{dy}{dx} = xy$ with $y(1) = 2$, take one step with $h = 0.2$ to estimate $y(1.2)$.

(b) For $\dfrac{dy}{dx} = x^2 + y$ with $y(0) = 0$, take two steps with $h = 0.5$ to estimate $y(1)$.

Accuracy and step length

33. This question uses the equation $\dfrac{dy}{dx} = y$ with $y(0) = 1$ from Question 31(b).

(a) Solve the equation exactly, and compare the exact $y(0.2)$ with your answer to Question 31(b).

(b) State whether Euler's method over- or under-estimates the solution, and explain why in terms of the shape of the curve.

(c) Estimate $y(0.2)$ again, using $h = 0.05$.

(d) Compare the errors of the two estimates, and state the effect of halving the step length.

Equilibrium values

34.

(a) A falling object has $\dfrac{dv}{dt} = 10 - 0.4v$, with v in $\mathrm{m\,s^{-1}}$. Find its terminal velocity.

(b) A skydiver falls from rest, with $\dfrac{dv}{dt} = 9.8 - 0.2v^2$ and v in $\mathrm{m\,s^{-1}}$. Find the terminal velocity.

(c) For the skydiver in part (b), use Euler's method with $h = 0.5$ to estimate $v(1)$.

(d) Explain why the estimate in part (c) cannot be accurate.

Concept check

35. A student says: "*Euler's method underestimates, because each step follows a tangent.*" Use Euler's method on $\dfrac{dy}{dx} = -2x$, $y(0) = 0$, with $h = 0.5$, to estimate $y(1)$. Hence show that the claim is false, and state when Euler's method does underestimate.

Chapter 16 · Differential Equations

Reflections

TOK Many laws of physics are differential equations: Newton's second law, radioactive decay, the spread of an epidemic. For the equations of this chapter, a rule for change together with one starting value determines the whole future of the system. This is mathematical determinism. Yet the same kind of equation can be so sensitive to its starting value that long-term prediction becomes impossible in practice, the phenomenon of *chaos*. Determined but unpredictable: what does that tell us about the limits of knowledge?

TOK A truncated series is deliberately wrong. It is chosen because it is simple enough to compute, and every extra term bought accuracy at the cost of that simplicity: four terms of the sine series give $\sin 1$ correct to five decimal places, and one term gives it to none.

Is there always a trade-off between accuracy and simplicity? And when a model is knowingly approximate, as every model in this chapter is, what makes it good enough to act on?

IM The series of this chapter came first from India. Mathematicians of the Kerala school, notably Madhava in the 14th century, found the expansions of $\sin x$, $\cos x$ and $\arctan x$ roughly three centuries before Newton, Leibniz and Maclaurin; the arctangent series, usually named after Gregory and Leibniz, was known in Kerala generations earlier. Numerical methods have a similar story. In the 15th century Jamshīd al-Kāshī, working at the observatory in Samarkand, needed $\sin 1^\circ$ to great precision for his astronomical tables. No formula gave it, so he wrote down an equation the value satisfied and solved it by repeated approximation, each round feeding its answer into the next, reaching sixteen correct decimal places. Neither idea was the work of one place or one name.

EXT The five standard series behave differently at the edges. The series for e^x, $\sin x$ and $\cos x$ converge for every x, but $\arctan x$ only for $|x| \leq 1$ and $\ln(1+x)$ only for $-1 < x \leq 1$, while the geometric series they are built from needs $|x| < 1$. Each series has a *radius of convergence*, and outside it the polynomial does not merely lose accuracy: it diverges completely. Finding that radius, and explaining why the radius is finite for $\ln(1+x)$ but infinite for e^x, is the beginning of complex analysis, where the answer depends on where the function is undefined in the complex plane. See Ch. 17.

Maclaurin series turn calculus into algebra: integrals with no elementary antiderivative, like $\int e^{-x^2}\,dx$, can be integrated term by term once the function is a series. Series also solve differential equations that no other method in this course can reach, by assuming the answer is a power series and matching coefficients. The infinite polynomial is a far-reaching idea in analysis.

End-of-Chapter Problems

1. GDC A population grows so that $\dfrac{dP}{dt} = 0.2P$, with $P = 500$ when $t = 0$.

(a) Solve the differential equation for $P(t)$. [4]

(b) Find the population when $t = 10$. [2]

2. GDC A body cools according to $\dfrac{d\theta}{dt} = -0.1(\theta - 25)$, with $\theta(0) = 90$ (in °C).

(a) Solve for $\theta(t)$. [4]

(b) Find the temperature after 10 minutes. [2]

(c) State the temperature the body approaches in the long run. [1]

3. Let $f(x) = \cos x$.

(a) Find $f(0), f'(0), f''(0), f'''(0), f^{(4)}(0)$. [3]

(b) Hence write the Maclaurin series of $\cos x$ up to the term in x^4. [2]

(c) Use it to estimate $\cos(0.2)$. [2]

4. EXACT

(a) Write down the Maclaurin series for $\sin x$ up to the term in x^5. [2]

(b) Hence evaluate $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x^3}$. [3]

5.

(a) Write down the Maclaurin series for e^x up to the term in x^3. [1]

(b) By substituting $-x^2$, find the series for e^{-x^2} up to x^4. [2]

(c) Hence estimate $\displaystyle\int_0^1 e^{-x^2}\, dx$ using terms up to x^4. [4]

6. Find the Maclaurin series of $\ln(1 - 3x)$ up to the term in x^3

(a) by substitution into a standard series; [3]

(b) by computing derivatives from the definition. [3]

7. By setting $x = 1$ in the Maclaurin series for $\arctan x$:

(a) write down a series for $\dfrac{\pi}{4}$; [2]

(b) estimate π using the first four terms; [2]

(c) comment on the rate of convergence. [1]

8. GDC A bowl of miso soup at $90°$C is left in a $25°$C room and obeys $\dfrac{d\theta}{dt} = -k(\theta - 25)$. After 10 minutes it has cooled to $65°$C.

(a) Solve the differential equation, expressing θ in terms of k and t. [4]
(b) Find k. [2]
(c) Find how long, from the start, the soup takes to reach $40°$C. [2]

9. Consider $\dfrac{dy}{dx} + \dfrac{1}{x}y = 3x$ for $x > 0$.

(a) Find the integrating factor. [2]
(b) Solve the equation given $y(1) = 2$. [4]

10. Consider $\dfrac{dy}{dx} = \dfrac{3x^2 + y^2}{2xy}$ for $x > 0$, $y > 0$.

(a) Use the substitution $y = vx$ to show that $x\dfrac{dv}{dx} = \dfrac{3 - v^2}{2v}$. [4]
(b) Hence find the general solution, giving y^2 in terms of x. [3]

11. Consider $\dfrac{dy}{dx} = \dfrac{y^2 + 3xy}{x^2}$ for $0 < x < \sqrt{3}$, with $y(1) = 1$.

(a) Use the substitution $y = vx$ to show that $x\dfrac{dv}{dx} = v^2 + 2v$. [3]
(b) Hence show that $y = \dfrac{2x^3}{3 - x^2}$. [5]

12. Solve each linear equation.

(a) $\dfrac{dy}{dx} - \dfrac{y}{x} = x^2$ for $x > 0$, given $y(1) = 1$. [6]
(b) $\dfrac{dy}{dx} + 2y = e^{-x}$ with $y(0) = 0$. [6]

13. Solve $\dfrac{dy}{dx} + (\tan x)\, y = \sec x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, given $y(0) = 2$. [7]

14. GDC Consider $\dfrac{dy}{dx} = y - 2x$, with $y(0) = 3$.

(a) Use Euler's method with $h = 0.25$ to estimate $y(0.5)$, showing each step. [4]
(b) Solve the equation exactly (integrating factor) and find the true $y(0.5)$. [4]
(c) Comment on the size and direction of the error. [2]

15. GDC Consider $\dfrac{dy}{dx} = y - x^2$ with $y(0) = 1$.

(a) Use Euler's method with $h = 0.25$ to estimate $y(0.5)$. [4]
(b) The exact solution is concave up on $0 \leq x \leq 0.5$. State, with a reason, whether your estimate is above or below the true value. [2]

16. GDC A rumour spreads through a school. The number of students who have heard it, P hundred, satisfies $\dfrac{dP}{dt} = P(4 - P)$, where t is the time in days and $P(0) = 1$.

(a) Use Euler's method with $h = 0.5$ to estimate $P(1.5)$, showing each step. [4]

(b) Find the non-zero value of P at which $\dfrac{dP}{dt} = 0$, and comment on the behaviour of the Euler estimates around it. [3]

17. GDC A fish population n (in thousands) in a lake satisfies the logistic equation $\dfrac{dn}{dt} = 0.4n(2 - n)$, where t is the time in years and $n(0) = 0.5$.

(a) Using partial fractions, show that the solution is $n = \dfrac{2}{1 + 3e^{-0.8t}}$. [6]

(b) Find the time at which the population reaches 1500 fish. [3]

18. EXACT Use Maclaurin series to evaluate $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{x \sin x}$. [5]

19. GDC

(a) Find the Maclaurin series of $e^{-x} \sin x$ up to the term in x^3. [4]

(b) Use your series to estimate $e^{-0.1} \sin 0.1$, and compare it with the value from your calculator. [3]

20. A function satisfies $\dfrac{dy}{dx} = x + y^2$ with $y(0) = 1$. Find the Maclaurin series of y up to the term in x^3. [7]

21. For each equation, name a method of solution, then find the general solution.

(a) $\dfrac{dy}{dx} = \dfrac{e^{2x}}{y}$, giving y^2 in terms of x [3]

(b) $\dfrac{dy}{dx} = \dfrac{x + 3y}{x}$, for $x > 0$ [4]

(c) $\dfrac{dy}{dx} + y \cot x = 2 \cos x$, for $0 < x < \pi$ [4]

22. GDC A drug is infused into the bloodstream at a rate that raises its concentration by r mg L^{-1} per hour, and the body removes it at a rate proportional to the concentration C, so that

$$\frac{dC}{dt} + kC = r,$$

where t is measured in hours and $C(0) = 0$.

(a) Explain why this equation is linear, and write down its integrating factor. [3]

(b) Solve the equation to show that $C = \dfrac{r}{k}\left(1 - e^{-kt}\right)$. [4]

(c) A patient is given $r = 6$ and has $k = 0.3$. Find the concentration after 4 hours. [2]

(d) Find the concentration the patient approaches in the long run, and explain what this means for the treatment. [3]

(e) Find how long it takes to reach 90% of that level. [2]

(f) Once that level is reached the infusion is stopped, so that $\dfrac{dC}{dt} = -kC$. Solve this equation and find how long the concentration then takes to halve. [3]

23. In this question you will build a series for logarithms that converges quickly.

(a) Find the Maclaurin series of $\ln(1 + x)$ up to the term in x^4. [4]

(b) Write down the Maclaurin series of $\ln(1 - x)$ up to the term in x^4. [2]

(c) Hence show that $\ln\left(\dfrac{1+x}{1-x}\right) = 2\left(x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \cdots\right)$. [3]

(d) By choosing a suitable value of x, use the three terms shown to estimate $\ln 3$. [3]

(e) The series for $\ln(1 + x)$ is valid only for $-1 < x \leq 1$. State the value of x that would be needed to reach $\ln 3$ from that series alone, and explain why the series in part (c) is the more useful one here. [2]

24. GDC Water drains from a tank through a hole in its base. The depth h metres at time t minutes satisfies

$$\frac{dh}{dt} = -k\sqrt{h},$$

where k is a positive constant.

(a) Separate the variables to show that $2\sqrt{h} = c - kt$ for some constant c. [3]

(b) Given that the depth is h_0 when $t = 0$, show that $h = \left(\sqrt{h_0} - \frac{1}{2}kt\right)^2$. [2]

(c) A tank starts with $h_0 = 1.44$ and has $k = 0.2$. Find how long it takes to empty. [3]

(d) Find the depth after 6 minutes. [2]

(e) Use Euler's method with steps of 3 minutes to estimate the depth after 6 minutes, and find the error in this estimate. [4]

(f) Describe the shape of the graph of h against t, and use it to explain why the Euler estimate lies below the exact value. [2]

Challenge Questions

25. GDC **Coffee in a room that warms and cools.** The chapter opened with a cup of coffee cooling in a room held at 20°. A real room is not held fixed. Suppose instead that the room's temperature is $20 + 5\sin\omega t$ degrees, where t is the time in minutes and $\omega > 0$ is a constant, so that the room moves between 15° and 25° once every $\frac{2\pi}{\omega}$ minutes. Newton's law of cooling then gives

$$\frac{d\theta}{dt} = -k\left(\theta - 20 - 5\sin\omega t\right),$$

where θ is the temperature of the coffee and $k > 0$ is a constant. The coffee is poured at 80°, so $\theta(0) = 80$.

(a) Let $u = \theta - 20$. Show that $\dfrac{du}{dt} + ku = 5k\sin\omega t$, and explain why this equation cannot be solved by separating the variables. [2]

(b) Using an integrating factor, and given that

$$\int e^{kt}\sin\omega t\,dt = \frac{e^{kt}\left(k\sin\omega t - \omega\cos\omega t\right)}{k^2+\omega^2} + C,$$

show that $u = \dfrac{5k\left(k\sin\omega t - \omega\cos\omega t\right)}{k^2+\omega^2} + Ce^{-kt}$, and find C. [5]

(c) Show that $k\sin\omega t - \omega\cos\omega t = \sqrt{k^2+\omega^2}\,\sin(\omega t - \alpha)$, where $\tan\alpha = \dfrac{\omega}{k}$ and $0 < \alpha < \frac{\pi}{2}$.

Hence show that, once t is large, $\theta \approx 20 + R\sin(\omega t - \alpha)$, where $R = \dfrac{5k}{\sqrt{k^2+\omega^2}}$, and explain why the coffee's temperature reaches each maximum $\frac{\alpha}{\omega}$ minutes after the room's. [5]

(d) Take $k = 0.05$. Find R and the delay $\frac{\alpha}{\omega}$ for a room that completes one cycle a day, $\omega = \frac{2\pi}{1440}$, and for a room whose heater switches on and off every 10 minutes, $\omega = \frac{2\pi}{10}$. [4]

(e) Using your answers to part (d), conjecture what happens to R and to the delay $\frac{\alpha}{\omega}$ as $\omega \to 0$ and as $\omega \to \infty$. Prove your conjectures for R, and use the Maclaurin series $\arctan z = z - \frac{z^3}{3} + \cdots$ to show that the delay tends to $\frac{1}{k}$ as $\omega \to 0$. Explain both results in terms of the coffee and the room. [4]

(f) A real room's temperature is not a single sine wave. Explain why, because the equation in part (a) is linear, a room temperature of 20 plus a sum of sine waves produces a long-run coffee temperature that is 20 plus the sum of the separate responses. Describe how you could use this to predict, and then test, the temperature of a drink left in a real room. [2]

26. GDC **How wrong is Euler?** Take $\dfrac{dy}{dx} = y$ with $y(0) = 1$, whose exact solution is $y = e^x$, so that $y(1) = e$.

(a) Show that one Euler step of length h gives $y_{n+1} = (1+h)y_n$, and hence that after n steps $y_n = (1+h)^n$. [3]

(b) Taking $h = \dfrac{1}{n}$, deduce that Euler's estimate of $y(1)$ is $\left(1 + \dfrac{1}{n}\right)^n$. [2]

(c) Evaluate this estimate for $n = 2, 4, 8, 16$, and tabulate the error in each case. [4]

(d) Describe what happens to the error each time h is halved, and state what this says about the accuracy of the method. [3]

(e) Use the Maclaurin series for $\ln(1+x)$ to show that, for $n \geq 2$,

$$n\ln\left(1 + \frac{1}{n}\right) = 1 - \frac{1}{2n} + \frac{1}{3n^2} - \cdots.$$

[3]

(f) Hence show that, for large n, the error $e - \left(1 + \dfrac{1}{n}\right)^n$ is approximately $\dfrac{e}{2n}$. Compare this with your table, and deduce the limit of $\left(1 + \dfrac{1}{n}\right)^n$ as $n \to \infty$. [3]

(g) The error is close to a known multiple of h. Suggest how two Euler estimates, one with

n steps and one with $2n$ steps, could be combined into a more accurate estimate, and test your suggestion on the values in your table. [3]

27. The ferry and the current. A river flows in the positive y-direction, and one bank is the line $x = 0$, with a dock at the origin. A ferry sets off from $(1, 0)$, moving through the water at a constant speed of 1 unit per minute and always aiming straight at the dock, while the current carries it in the positive y-direction at k units per minute, where $k > 0$. Its velocity therefore has components

$$\frac{dx}{dt} = -\frac{x}{\sqrt{x^2+y^2}}, \qquad \frac{dy}{dt} = -\frac{y}{\sqrt{x^2+y^2}} + k,$$

and its path satisfies

$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2+y^2}}{x}, \qquad x > 0.$$

(a) Show the equation is homogeneous, and that $y = vx$ leads to $x\dfrac{dv}{dx} = -k\sqrt{1+v^2}$. [3]

(b) Given that $\displaystyle\int \frac{dv}{\sqrt{1+v^2}} = \ln\left(v + \sqrt{1+v^2}\right) + C$, show that $v + \sqrt{1+v^2} = x^{-k}$. [3]

(c) Hence show that $y = \frac{1}{2}\left(x^{1-k} - x^{1+k}\right)$. [4]

(d) Taking $k = \frac{1}{2}$, find the point that the ferry approaches as $x \to 0$. [2]

(e) Show that, as $x \to 0$, the ferry approaches the dock if and only if $k < 1$. Describe what happens to the path when $k = 1$ and when $k > 1$. [4]

(f) Using part (b), show that $\sqrt{1+v^2} = \frac{1}{2}\left(x^{-k} + x^k\right)$. Hence, using the expression for $\dfrac{dx}{dt}$, show that for $0 < k < 1$ the ferry reaches the dock after

$$T = \frac{1}{2}\int_0^1 \left(x^{-k} + x^k\right) dx = \frac{1}{1-k^2} \text{ minutes,}$$

and explain what the integral shows when $k \geq 1$. [5]

(g) Aiming at the dock is not the only way to steer. Suggest a different strategy for $0 < k < 1$, find the time it takes, and compare it with T. [2]

28. GDC **The series for $\tan x$.** In the notes, the equation $\dfrac{dy}{dx} = 1 + y^2$ with $y(0) = 0$ gave the first two terms of the Maclaurin series of its solution, $y = \tan x$, by repeated differentiation. This question finds more terms by a quicker method and asks how far the series can be trusted. You may assume that $y = \tan x$ is the only solution with $y(0) = 0$ on $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and that it is equal to its Maclaurin series near $x = 0$.

(a) Show that if $y = g(x)$ satisfies $\dfrac{dy}{dx} = 1 + y^2$ with $g(0) = 0$, then so does $y = -g(-x)$. Deduce that $\tan x$ is an odd function, so that its series contains only odd powers of x. [3]

(b) Write $y = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + \cdots$. Substitute into the equation and compare the coefficients of x^0, x^2 and x^4 to show that $a_1 = 1$, $a_3 = \frac{1}{3}$ and $a_5 = \frac{2}{15}$. Find a_7. [6]

(c) Use the four terms found in part (b) to estimate $\tan 0.5$, $\tan 1$ and $\tan 1.5$, and compare each estimate with the value from your calculator. [3]

(d) Using the graph of $y = \tan x$, explain why no polynomial can stay close to $\tan x$ as $x \to \frac{\pi}{2}$, and suggest the largest interval centred on 0 on which the series could converge. [3]

(e) Apply the method of part (b) to $\dfrac{dy}{dx} = 1 - y^2$ with $y(0) = 0$, as far as the term in x^7. Conjecture how its coefficients are related to those of $\tan x$, and explain why the relationship continues for every power. [3]

(f) The coefficients $1, \frac{1}{3}, \frac{2}{15}, \frac{17}{315}, \frac{62}{2835}, \ldots$ follow no obvious pattern. Investigate the ratio of successive coefficients, and suggest how its behaviour is connected to your answer to part (d). [2]