CALCULUS

Chapter 15

Integral Calculus

15

SYLLABUS 5.5 5.9 5.10 5.11 5.15 5.16 5.17

In Ch. 13 a speedometer reading came from distance: the speed was the rate at which distance changed. Now reverse the question. Suppose you record the speedometer for an entire journey. Can you work out how far you travelled, using only the speed?

Over a tiny slice of time, your speed barely changes, so the distance covered in that slice is roughly speed times the slice of time. Add up all the slices and you have the total distance. As the slices become narrower, the rough sum approaches the exact distance. What you are computing is the area under the speed-time graph.

This is the second main idea of calculus, and it has two parts. The first is **reversing a derivative**: given a rate of change, recover the quantity it came from. The second is **accumulating**: adding up a great many thin slices to get a total. The area under a graph is exactly this kind of total.

They have no obvious connection. Reversing a derivative is a problem in algebra. Adding slices is a problem in geometry. Yet the car journey links them, because the distance travelled is both the reverse of the velocity and the area under the velocity graph.

The chapter is built in that order. Section 15.1 does the reversal on its own, with no mention of area. Section 15.2 defines the area on its own, with no mention of derivatives, and then proves that the two are the same calculation. That result is the Fundamental Theorem of Calculus. By the end of this chapter you will be able to find antiderivatives of the standard functions, evaluate definite integrals using the Fundamental Theorem, integrate by substitution and by parts, find areas between curves and volumes of revolution, and find displacement and distance travelled from velocity.

15.1 Antiderivatives

This chapter reverses differentiation. Given a function f, the task is to find a function whose derivative is f. Such a function is called an **antiderivative** of f.

An antiderivative is never unique. The derivative of a constant is zero, so any constant may be added to an antiderivative without altering its derivative. Antiderivatives therefore occur in families whose members differ by a constant.

DEF F is an **antiderivative** of f when $F'(x) = f(x)$. If F is one antiderivative, then $F(x) + C$ is an antiderivative for every constant C, and on an interval these are all of them.

Naming the whole family in words each time is long, so it is given a symbol of its own.

KEY

$$\int f(x)\,dx = F(x) + C \qquad \text{means} \qquad F'(x) = f(x).$$

This is the **indefinite integral** of f, and C is the **constant of integration**. It is easy to forget, and an answer written without it names a single antiderivative rather than the whole family.

Reversing the power rule means raising the power by one and dividing by the new power.

KEY · IB

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

The exception $n = -1$ is handled in §15.3. Constant multiples and sums are reversed term by term, just as they are differentiated term by term.

Worked through. To find $\int (3x^2 - 4x + 1)\,dx$, reverse each term, raising each power by one:

$$\int (3x^2 - 4x + 1)\,dx = x^3 - 2x^2 + x + C.$$

Check by differentiating: $\frac{d}{dx}(x^3 - 2x^2 + x + C) = 3x^2 - 4x + 1$. ✓ Differentiating the answer verifies it completely.

15.1.1 Boundary conditions

The $+C$ means that an antiderivative is not determined until further information is supplied. A single condition, such as a point through which the curve passes, fixes C and selects one member of the family.

Worked through. To find the equation of a curve with gradient $\dfrac{dy}{dx} = 6x^2 - 2$ that passes through $(1, 4)$, antidifferentiate to recover y:

$$y = 2x^3 - 2x + C.$$

Use the point $(1, 4)$: $4 = 2 - 2 + C \implies C = 4$. So $y = 2x^3 - 2x + 4$.

YOUR TURN 15.1 Antiderivatives

Integrating term by term

1. Find each indefinite integral.

(a) $\displaystyle\int (4x^3 - 6x)\,dx$

(b) $\displaystyle\int (x^2 + 3x - 5)\,dx$

(c) $\displaystyle\int \frac{6}{x^3}\,dx$

(d) $\displaystyle\int \sqrt{x}\,dx$

(e) $\displaystyle\int \left(x^2 - \frac{1}{x^2}\right)dx$

(f) $\displaystyle\int \frac{x^3 - 2x}{x}\,dx$

(g) $\displaystyle\int x^{-1/2}\,dx$

(h) $\displaystyle\int (2x)^3\,dx$

Using a condition to find the constant

2.

(a) A curve has $\dfrac{dy}{dx} = 4x - 3$ and passes through $(2, 5)$. Find y.

(b) A curve has $\dfrac{dy}{dx} = 3\sqrt{x}$ and passes through $(4, 20)$. Find y.

(c) A curve has $\dfrac{d^2y}{dx^2} = 6x$, with $\dfrac{dy}{dx} = 1$ and $y = 2$ when $x = 0$. Find y. *(Hint: integrate twice, and use one condition after each integration)*

(d) A curve has $\dfrac{d^2y}{dx^2} = 12x^2 - 4$. Its gradient is 0 at the point $(1, 3)$. Find y.

(e) Explain why a gradient function alone does not determine a curve, and what extra information does.

(f) Explain why parts (c) and (d) needed two pieces of information rather than one.

Concept check

3. A student writes $\displaystyle\int \frac{1}{x^2}\,dx = \frac{1}{x^3/3} + C = \frac{3}{x^3} + C$. Identify the error, and find the correct integral.

15.2 Definite Integrals and Area

Set antiderivatives aside for the moment. This section takes up a second and apparently unrelated problem; the two are connected only later.

The problem is area. Regions with straight edges, and a few curved shapes such as the circle, have area formulas. A region under a general curve has none. The available approach is approximation.

Cut the interval from a to b into n strips, each of width Δx. Replace each strip by a rectangle whose height is the value of the function somewhere in that strip. Every rectangle has area height $\times$ width, so the total is

$$\sum_{i=1}^{n} f(x_i)\,\Delta x.$$

This is not the area. It is a sum of rectangles, and it differs from the area by whatever those rectangles fail to cover or cover in excess. Narrower strips leave less room for that error, and as n increases the sum approaches a single value.

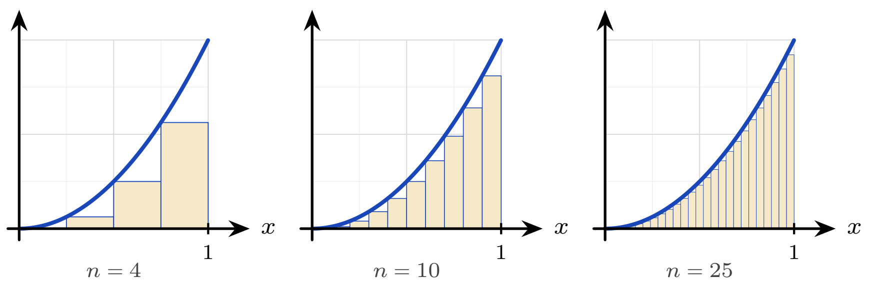

Figure 15.1 Rectangles under $y = x^2$ on $[0, 1]$, taking the height at the left of each strip. Four strips give 0.219, ten give 0.285 and twenty-five give 0.314; the true area is $\frac{1}{3}$. The missing pieces above each rectangle shrink as the strips narrow.

The definition allows the height to be read anywhere in the strip, but two choices are natural: the value at the left end, or the value at the right. Taking both is what makes the argument work, because one sum is too small and the other too large.

For a curve that rises across the interval, as $y = x^2$ does on $[0, 1]$, every left-hand rectangle falls short of the curve and every right-hand one overshoots it. The first sum therefore underestimates the area and the second overestimates it, so the true value lies between them. For a curve that falls across the interval the two exchange roles, and the bracketing still holds.

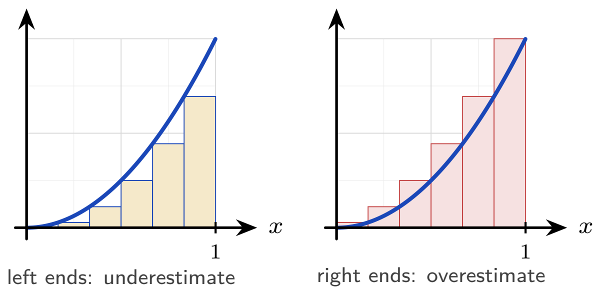

Figure 15.2 The same six strips, with the height of each rectangle read at the left of its strip and then at the right. On a rising curve the left-hand rectangles all sit below it and the right-hand ones all rise above it, so the area lies between the two sums. Narrowing the strips closes the gap from both sides at once.

The numbers behave as the picture predicts.

n	4	10	100	1000
low estimate	0.219	0.285	0.328	0.3328
high estimate	0.469	0.385	0.338	0.3338

The two rows converge, and the value they enclose is $\frac{1}{3}$. That limit is what is meant by the area, and it is called the **definite integral**.

DEF The **definite integral** of f from a to b is the limit of the rectangle sums as the strips become arbitrarily narrow:

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_i f(x_i)\,\Delta x.$$

For $f(x) \geq 0$ it is the area between the curve and the x-axis from a to b.

The notation can now be read directly. The elongated S denotes a sum, $f(x)\,dx$ represents a single rectangle of height $f(x)$ and very small width dx, and the integral accumulates them.

As it stands, however, the definition gives no practical way to calculate an area. Nothing in it indicates how such a limit is to be evaluated, and summing a thousand rectangles by hand is not a practical method.

The **Fundamental Theorem of Calculus** supplies the missing method. It states that this limit of sums can be evaluated by antidifferentiation, so the two ideas of the chapter are in fact one calculation.

KEY **The Fundamental Theorem of Calculus.** If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

Reversing a derivative and adding up rectangles give the same answer. Test it on the area the table was closing in on. An antiderivative of x^2 is $\frac{x^3}{3}$, so

$$\int_0^1 x^2\,dx = \left[\tfrac{x^3}{3}\right]_0^1 = \tfrac{1}{3} - 0 = \tfrac{1}{3},$$

the value the thousand rectangles were approaching, found in one line. The constant C cancels in the subtraction, which is why a definite integral is a plain number while an indefinite one carries a $+C$.

The theorem can also be read the other way round. Write the integrand as a derivative, $f = F'$:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Adding up a rate of change across an interval gives the total change over that interval: the area under the graph of F' equals the rise of the graph of F. This is the reading used in §15.9, where the area under a velocity graph is a displacement.

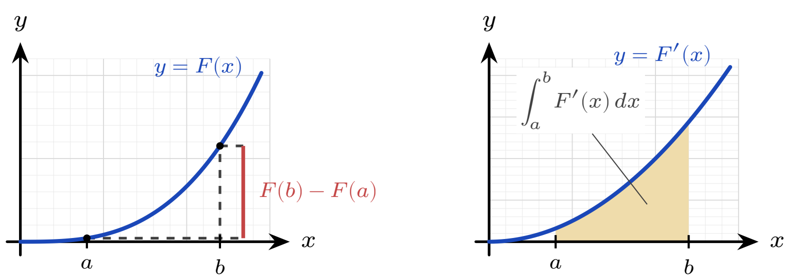

Figure 15.3 Left, $F(x) = \frac{x^3}{3}$ rises by $F(b) - F(a)$ between $x = a$ and $x = b$. Right, its derivative $F'(x) = x^2$, with the area over the same interval shaded. The rise on the left and the area on the right are the same number: for $a = 0.4$ and $b = 1.2$ both are 0.555 to 3 s.f.

The Fundamental Theorem should not be accepted on trust. Its proof uses only the definition of the derivative and one picture of a thin strip, and it shows exactly how the area under a curve and the rate of change are connected. It is given at the end of this chapter, after the Reflections. Read it once you can use the theorem: the proof is short, and it explains why the two halves of this chapter are one idea.

KEY · IB

$$A = \int_a^b y\,dx \qquad (y \geq 0)$$

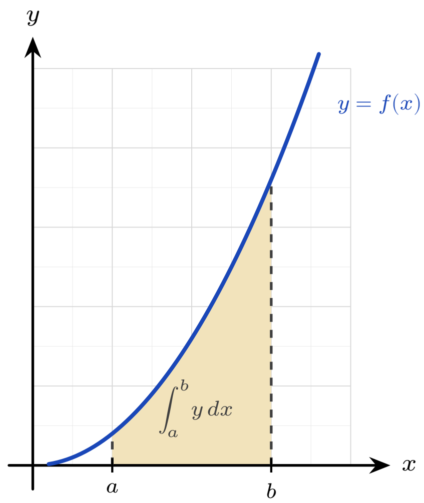

Figure 15.4 For $y \geq 0$ the definite integral measures the shaded area between the curve and the x-axis, from $x = a$ to $x = b$.

EXAMPLE 15.1

Evaluate $\displaystyle\int_0^2 3x^2\,dx$.

$$\int_0^2 3x^2\,dx = \left[x^3\right]_0^2 = 8 - 0 = 8.$$

TIP Your GDC evaluates definite integrals directly, and on a calculator paper that is often all a question requires. Even when analytic working is demanded, the GDC value is a free check of your antiderivative.

15.2.1 Areas below the axis

Where the curve goes below the x-axis, the integral is negative: it is a signed area. For a true geometric area you integrate the parts separately and take the absolute value of any negative piece.

KEY · IB

$$A = \int_a^b |y|\,dx$$

Without a calculator, find where $y = 0$ between a and b, integrate over each piece separately, and add the absolute values of the results. With a GDC, evaluate $\int_a^b |y|\,dx$ directly.

The distinction is visible as soon as a curve crosses the axis more than once.

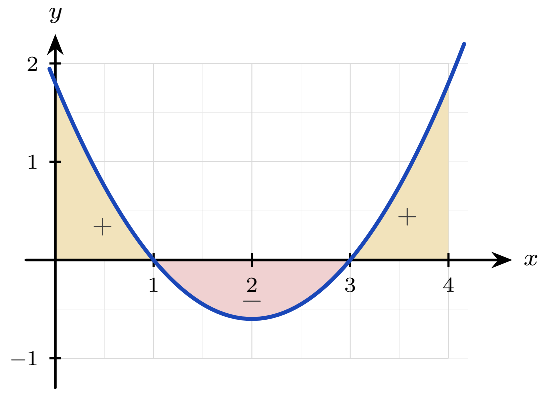

Figure 15.5 A definite integral accumulates *signed* area. The two gold regions lie above the axis and count positively; the scarlet region lies below and counts negatively. A single integral across the whole interval returns the sum of the three signed pieces, which is smaller than the total area enclosed.

EXAMPLE 15.2

Find the area enclosed between $y = x^2 - 4$ and the x-axis from $x = 0$ to $x = 2$.

On $[0, 2]$ the curve lies below the axis, so the integral is negative:

$$\int_0^2 (x^2 - 4)\,dx = \left[\tfrac{x^3}{3} - 4x\right]_0^2 = \tfrac{8}{3} - 8 = -\tfrac{16}{3}.$$

The area is the magnitude, $\dfrac{16}{3}$. An area is never negative, so $-\frac{16}{3}$ answers a different question from the one asked.

Worked through. To find the area enclosed between $y = x^2 - 1$ and the x-axis from $x = 0$ to $x = 2$, split the interval where the curve crosses the axis. The curve crosses the axis at $x = 1$: it is below the axis on $(0, 1)$ and above it on $(1, 2)$. Split there:

$$\int_0^1 (x^2 - 1)\,dx = \left[\tfrac{x^3}{3} - x\right]_0^1 = -\tfrac{2}{3}, \qquad \int_1^2 (x^2 - 1)\,dx = \left[\tfrac{x^3}{3} - x\right]_1^2 = \tfrac{4}{3}.$$

The area is $\left|-\frac{2}{3}\right| + \frac{4}{3} = 2$. A single integral from 0 to 2 gives $-\frac{2}{3} + \frac{4}{3} = \frac{2}{3}$, because the piece below the axis cancels part of the piece above.

CAUTION Before finding an area, check where the curve crosses the x-axis. A curve that goes above and below will have positive and negative parts that partly cancel in a single integral, giving the wrong area. Split at the roots.

YOUR TURN 15.2 Definite Integrals and Area

Evaluating definite integrals

4. Evaluate each definite integral.

(a) $\displaystyle\int_1^2 4x\,dx$

(b) $\displaystyle\int_0^3 (x^2+1)\,dx$

(c) $\displaystyle\int_1^2 \frac{6}{x^3}\,dx$

(d) $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\,dx$

(e) $\displaystyle\int_0^2 (x+1)(x-3)\,dx$

(f) $\displaystyle\int_{-1}^2 (3x^2-2x)\,dx$

Areas and the x-axis

5. Each region lies on one side of the x-axis. Find its area.

(a) The region under $y = x^2$ between $x = 1$ and $x = 3$.

(b) The region enclosed between $y = x^2 - 1$ and the x-axis.

6. Each curve crosses the x-axis inside the interval. Find the total area between the curve and the x-axis over the interval given.

(a) $y = x^3$, from $x = -2$ to $x = 2$.

(b) $y = x^2 - x - 2$, from $x = 0$ to $x = 3$. State also the value of $\displaystyle\int_0^3 (x^2-x-2)\,dx$, and explain why it is not the area.

(c) $y = x^3 - x$, from $x = 0$ to $x = 2$.

Estimating an area with rectangles

7. The area under $y = x^2$ from $x = 0$ to $x = 2$ is to be estimated by four rectangles of equal width.

(a) Taking the height of each rectangle at the left of its strip, show that the estimate is 1.75.

(b) Taking the height at the right instead, show that the estimate is 3.75.

(c) Find the exact area, and confirm that it lies between the two estimates.

(d) Repeat part (a) with eight rectangles, and show that the left-hand estimate rises to 2.1875.

(e) State what happens to the two estimates as the number of rectangles increases.

Concept check

8. A student writes

$$\int_1^3 \frac{2}{x^3}\,dx = \left[-\frac{1}{x^2}\right]_1^3 = -\frac{1}{9} - 1 = -\frac{10}{9}.$$

Explain why a negative answer cannot be right here, identify the error, and find the correct value.

15.3 Standard Integrals

Every derivative established in Ch. 13 can be read in reverse. Doing so converts the table of derivatives into a table of integrals, and turns the question "what is the integral of this function?" into the easier question "what was differentiated to produce it?"

KEY · IB

$$\int \frac{1}{x}\,dx = \ln|x| + C, \qquad \int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C, \qquad \int e^x\,dx = e^x + C$$

None of these is a new fact. Each is a derivative already known, read from right to left, and each should be understood rather than memorised.

- $\int \frac{1}{x}\,dx = \ln|x| + C$ fills the one gap the power rule leaves. Raising $n = -1$ by one gives 0, and the rule then asks you to divide by that new power, which is impossible. So the antiderivative of x^{-1} cannot be a power at all. The logarithm supplies it, because $\frac{d}{dx}\ln x = \frac{1}{x}$. The modulus is there because $\frac{1}{x}$ is defined for negative x as well, and on that side $\frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}$ by the chain rule: the same derivative on both sides of the origin.
- $\int e^x\,dx = e^x + C$ because e^x is equal to its own derivative. A function that is unchanged by differentiating is unchanged by antidifferentiating.
- $\int \cos x\,dx = \sin x + C$ is the direct reversal of $\frac{d}{dx}\sin x = \cos x$, with no sign to watch.
- $\int \sin x\,dx = -\cos x + C$ carries a minus sign that is easily dropped. Differentiating cosine introduces one: $\frac{d}{dx}\cos x = -\sin x$. To finish with $+\sin x$ you must therefore start from $-\cos x$.

The two trigonometric cases are easiest to remember as a cycle. Differentiating moves one step clockwise; integrating moves one step back.

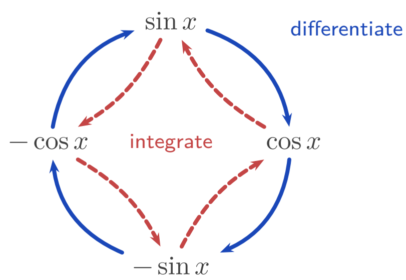

Figure 15.6 The four functions form a closed cycle. Differentiating advances one step clockwise (solid), integrating retreats one step anticlockwise (dashed). Reading the cycle backwards from $\sin x$ lands on $-\cos x$, which is where the minus sign in $\int \sin x\, dx$ comes from.

At HL, three more standard integrals are provided, the reverses of the exponential and inverse-trigonometric derivatives.

KEY · IB HL

$$\int a^x\, dx = \frac{a^x}{\ln a} + C, \qquad \int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\arctan\frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \arcsin\frac{x}{a} + C, \qquad |x| < a$$

These three reward the same reading. Differentiating a^x produces an extra factor: writing $a^x = e^{x\ln a}$ and applying the chain rule gives $\frac{d}{dx}a^x = a^x \ln a$. Dividing by $\ln a$ removes it. When $a = e$ the factor $\ln e$ is 1 and the formula becomes $\int e^x dx = e^x + C$, as it must.

The two inverse-trigonometric forms come from $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$ and $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$, with a acting as a scale. Their asymmetry is a common source of error: the arctangent form carries a factor $\frac{1}{a}$ and the arcsine form does not. In each case differentiating x/a contributes a chain-rule factor $\frac{1}{a}$. For the arcsine, $\frac{d}{dx}\arcsin\frac{x}{a} = \frac{1}{\sqrt{a^2-x^2}}$: the $\sqrt{a^2} = a$ that comes out of the square root cancels the $\frac{1}{a}$ exactly. For the arctangent, $\frac{d}{dx}\arctan\frac{x}{a} = \frac{a}{a^2+x^2}$: the denominator contributes a^2, which more than cancels the $\frac{1}{a}$ and leaves a factor a, so the formula needs $\frac{1}{a}$ in front to remove it.

Reading the rest of the HL derivative table backwards gives four more. They must be learned, because the formula booklet gives only the derivatives:

KEY HL

$$\int \sec^2 x\, dx = \tan x + C, \qquad \int \sec x \tan x\, dx = \sec x + C,$$

$$\int \operatorname{cosec}^2 x\, dx = -\cot x + C, \qquad \int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$$

EXAMPLE 15.3
Find (a) $\displaystyle\int \frac{1}{x^2+25}\,dx$, (b) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$.
Match the standard forms with $a = 5$ and $a = 3$:

$$\text{(a) } \tfrac{1}{5}\arctan\frac{x}{5} + C, \qquad \text{(b) } \arcsin\frac{x}{3} + C.$$

The only work is recognising a^2 as 25 and as 9. After that the standard form gives the answer.

Definite integrals of these standard forms are evaluated exactly as in §15.2: find the antiderivative, then substitute the limits.

Worked through. The antiderivative of $\sin x$ is $-\cos x$, so

$$\int_0^{\pi/2} \sin x\,dx = \Big[-\cos x\Big]_0^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = 0 + 1 = 1.$$

The area under one quarter-wave of the sine curve is exactly 1, with no π in the answer. In the same way, the antiderivative of $\dfrac{1}{x}$ is $\ln|x|$, and when both limits are positive

$$\int_1^{e^2} \frac{1}{x}\,dx = \Big[\ln x\Big]_1^{e^2} = \ln e^2 - \ln 1 = 2 - 0 = 2.$$

Stopping at $x = e$ instead would give an area of exactly one. This gives one way to define e: it is the upper limit that makes the area under $y = \frac{1}{x}$, starting at $x = 1$, equal to one.

15.3.1 Linear composites

Sometimes the function inside is linear, as in e^{2x} or $\cos 3x$. Integrate as usual, then divide by the coefficient of x inside. This reverses the chain rule, whose extra factor was that same coefficient.

KEY

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C, \qquad \text{where } F' = f$$

EXAMPLE 15.4

Find (a) $\int e^{2x}\,dx$, (b) $\int (2x+1)^5\,dx$, (c) $\int \cos 3x\,dx$.
Each has a linear function inside, so integrate and then divide by the coefficient of x:

$$\text{(a) } \tfrac{1}{2}e^{2x} + C, \qquad \text{(b) } \frac{(2x+1)^6}{12} + C, \qquad \text{(c) } \tfrac{1}{3}\sin 3x + C.$$

In (b), reversing the power rule gives $\frac{(2x+1)^6}{6}$, then dividing by the coefficient 2 gives the 12.

The linear-composite rule extends to the whole table. For instance, $\int \sec^2(2x+5)\,dx = \frac{1}{2}\tan(2x+5) + C$: integrate as usual, then divide by the inner coefficient.

YOUR TURN 15.3 Standard Integrals

Reversing the standard derivatives

9. Find each indefinite integral.

(a) $\int (2\cos x - \sin x)\,dx$

(b) $\int \left(3e^x + \frac{1}{x}\right) dx$

Linear composites

10. Find each indefinite integral.

(a) $\int e^{3x}\,dx$

(b) $\int (3x-2)^4\,dx$

(c) $\int \sin 2x\,dx$

(d) $\int \sec^2(4x)\,dx$

(e) $\int \cos(3x+1)\,dx$

(f) $\int \frac{1}{2x+5}\,dx$

The HL standard forms

11. Find each indefinite integral, matching it to a standard form.

(a) $\int \frac{1}{x^2+9}\,dx$

(b) $\int \frac{1}{\sqrt{16-x^2}}\,dx$

(c) $\int 5^x\,dx$

(d) $\int \frac{2}{4+x^2}\,dx$

(e) $\int \frac{1}{\sqrt{9-4x^2}}\,dx$

12. Find each indefinite integral by reading a derivative in reverse.

(a) $\int \operatorname{cosec}^2 x\,dx$

(b) $\int \sec x \tan x\,dx$

(c) $\int 3\operatorname{cosec} 2x \cot 2x\,dx$

Definite integrals of standard forms

13.

(a) Evaluate $\displaystyle\int_1^{e^3} \frac{2}{x}\,dx$.

(b) Evaluate $\displaystyle\int_0^{\pi/6} \cos x\,dx$.

(c) Find the area under $y = \sin x$ from $x = 0$ to $x = \pi$.

(d) Evaluate $\displaystyle\int_0^{2\pi} \sin x\,dx$, and explain why the answer is not the area between the curve and the axis over that interval.

Concept check

14. A student writes $\displaystyle\int e^{x^2}\,dx = \frac{e^{x^2}}{2x} + C$, dividing by the derivative of the inner function as for $\displaystyle\int e^{3x}\,dx = \tfrac{1}{3}e^{3x} + C$. Show, by differentiating, that the answer is wrong, and explain why the method works for e^{3x} but not for e^{x^2}.

15.4 Rewriting Before Integrating HL

Many integrands do not match a standard form as they stand, but can be rewritten until they do. Four rewritings cover the cases met in this course: divide, complete the square, split into partial fractions, or replace a squared trigonometric function using a double angle. For the first three, deciding between them takes one look at the denominator. The table in §15.6.2 summarises how to choose between all the methods of this chapter.

15.4.1 Divide first

When the numerator has degree at least that of the denominator, the fraction is **improper** and no standard form applies. Divide before anything else. For instance $\dfrac{x^2}{x+1} = x - 1 + \dfrac{1}{x+1}$, so the integral is $\frac{x^2}{2} - x + \ln|x+1| + C$. Only when the numerator has the lower degree are the next two methods available.

15.4.2 Completing the square HL

A denominator does not always arrive in the form $a^2 + x^2$. A quadratic such as $x^2 + 2x + 5$ has that shape, but only after you complete the square.

Worked through. To find $\displaystyle\int \frac{1}{x^2 + 2x + 5}\,dx$, complete the square: $x^2 + 2x + 5 = (x+1)^2 + 4$. This is the arctangent shape with $a = 2$ and the shifted variable

$x+1$:

$$\int \frac{dx}{(x+1)^2+4} = \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C.$$

Any quadratic x^2+bx+c with no real roots can be written in the form a^2+u^2 this way. The shift from x to $x+1$ does not change the integral, because the derivative of $x+1$ is 1.

15.4.3 Partial fractions HL

When the quadratic *does* factorise, use a different method. Split the fraction into partial fractions, as in Ch. 6. Each piece then has the form $\frac{1}{x+k}$, and integrates to a logarithm.

Worked through. To find $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$, factor and decompose: $\displaystyle\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$. Then

$$\int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx = \ln|x+1| - \ln|x+2| + C = \ln\left|\frac{x+1}{x+2}\right| + C.$$

So the denominator decides the method. If it factorises, use partial fractions, which lead to logarithms. If it has no real roots, complete the square, which leads to an arctangent.

15.4.4 Using a double angle

There is no standard integral for $\sin^2 x$ or $\cos^2 x$, and neither is the derivative of anything simple. The method is to remove the square before integrating, using the double-angle identity from Ch. 8:

$$\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1.$$

Rearranged, these give the two rewritings that matter here:

KEY

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

Each right-hand side is a constant plus a linear composite, and both of those integrate immediately.

Worked through. To find $\displaystyle\int \sin^2 x\,dx$, and then $\displaystyle\int_0^{\pi} \sin^2 x\,dx$, replace the square first:

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

The $\frac{1}{4}$ comes from the linear composite: integrating $\cos 2x$ gives $\frac{1}{2}\sin 2x$, and the outer $\frac{1}{2}$ halves it again.

For the definite integral, now use the limits:

$$\int_0^{\pi} \sin^2 x\,dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi} = \left(\frac{\pi}{2} - 0\right) - (0 - 0) = \frac{\pi}{2}.$$

The answer comes out cleanly, and there is a reason for that. Over a full arch, $\sin^2 x$ averages exactly $\frac{1}{2}$, so the area is half the width π. The oscillating part, $\sin 2x$, contributes nothing over a whole number of half-periods.

The same rewriting handles $\cos^2 x$, and it is the standard way to integrate the square of a sine or cosine. Expect it when a volume of revolution involves a sine or cosine curve, since the volume formula squares the function.

YOUR TURN 15.4 Rewriting Before Integrating HL

Dividing first

15. Rewrite each integrand before integrating.

(a) $\displaystyle\int \frac{x^2 + 3x}{x}\,dx$

(b) $\displaystyle\int \frac{2x + 1}{x}\,dx$

(c) $\displaystyle\int \frac{x^3 - 1}{x^2}\,dx$

(d) $\displaystyle\int \frac{2x + 3}{x + 1}\,dx$

(e) $\displaystyle\int \frac{x^2 + 1}{x + 1}\,dx$

Partial fractions

16.

(a) Find $\displaystyle\int \frac{1}{x^2 - 9}\,dx$ using partial fractions.

(b) Find $\displaystyle\int \frac{4x + 5}{(x + 1)(x + 2)}\,dx$.

Completing the square

17.

(a) Find $\displaystyle\int \frac{1}{x^2 + 4x + 13}\,dx$.

(b) Find $\displaystyle\int \frac{1}{2x^2 + 4x + 10}\,dx$.

(c) Explain how you decide, from the denominator alone, whether to use partial fractions or to complete the square.

Using a double angle

18.

(a) Find $\displaystyle\int \cos^2 x\,dx$.

(b) Find $\displaystyle\int \sin^2 3x\,dx$.

(c) Evaluate $\displaystyle\int_0^{\pi/6} \sin^2 x\,dx$, giving an exact answer.

Concept check

19. A student writes $\displaystyle\int \sin^2 x\,dx = \frac{\sin^3 x}{3} + C$. Identify the error, and find the correct integral.

20. A student writes $\displaystyle\int \frac{1}{x^2+4}\,dx = \ln\left(x^2+4\right) + C$. Identify the error, and find the correct integral.

15.5 Integration by Substitution

Substitution reverses the chain rule. Look for a composite function whose inner function also appears, up to a constant multiple, as a factor outside. Put u equal to that inner function, and a hard integral in x becomes an easy one in u.

The method has four steps.

1. Let u be the inner function.
2. Replace dx using $du = \dfrac{du}{dx}\,dx$, so that no x remains.
3. Integrate with respect to u.
4. Substitute back to write the answer in terms of x.

Worked through. In $\int 2x(x^2+1)^3\,dx$, let $u = x^2+1$, so $\dfrac{du}{dx} = 2x$, giving $du = 2x\,dx$. The integral becomes

$$\int u^3\,du = \frac{u^4}{4} + C = \frac{(x^2+1)^4}{4} + C.$$

The factor $2x$ outside is exactly the $\frac{du}{dx}$ required. Without it the substitution would leave an x behind and fail.

EXAMPLE 15.5

Find $\displaystyle\int \frac{2x}{x^2+1}\,dx$.

With $u = x^2 + 1$, $du = 2x\,dx$:

$$\int \frac{1}{u}\,du = \ln|u| + C = \ln(x^2+1) + C.$$

The numerator is the derivative of the denominator, and that is why the substitution works.

The same substitution, $u = f(x)$, works for every fraction whose numerator is the derivative of its denominator:

KEY

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

CAUTION For a definite integral by substitution, either convert the limits to u-values, or substitute back to x before applying the original limits. Applying the x-limits to a u-expression gives a wrong answer.

EXAMPLE 15.6

Evaluate $\displaystyle\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx$.

Let $u = x^2 + 1$, so $du = 2x\,dx$. Convert the limits too: $x = 0 \Rightarrow u = 1$ and $x = \sqrt{3} \Rightarrow u = 4$. Then

$$\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx = \frac{1}{2}\int_1^4 \sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}\left[u^{3/2}\right]_1^4 = \frac{1}{3}(8-1) = \frac{7}{3}.$$

Once the limits are converted, you never return to x at all; the whole calculation finishes in u.

Substitution is also the method for many exponential and trigonometric integrands that match no standard form.

EXAMPLE 15.7

Find (a) $\displaystyle\int xe^{-x^2}\,dx$, (b) $\displaystyle\int \sin^4 x\cos x\,dx$.

(a) The inner function is $-x^2$, and its derivative $-2x$ appears outside up to the constant -2. Let $u = -x^2$, so $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}\,du$:

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C.$$

On its own, e^{-x^2} cannot be integrated in terms of the functions of this course. The factor x is what makes the integral possible, and without it these methods cannot find the integral.

(b) Here the inner function is $\sin x$, and its derivative $\cos x$ is sitting outside. Let $u = \sin x$, so $du = \cos x\,dx$:

$$\int \sin^4 x\cos x\,dx = \int u^4\,du = \frac{u^5}{5} + C = \frac{\sin^5 x}{5} + C.$$

In both parts the test is the same: is the derivative of the inner function present as a factor? If it is, substitution usually works. If it is not, another method is usually quicker.

YOUR TURN 15.5 Integration by Substitution

Indefinite integrals by substitution

21. Find each integral by substitution.

(a) $\displaystyle\int 3x^2(x^3+1)^4\,dx$

(b) $\displaystyle\int xe^{x^2}\,dx$

(c) $\displaystyle\int 2x\sqrt{x^2+1}\,dx$

(d) $\displaystyle\int \frac{x}{(x^2+4)^2}\,dx$

(e) $\displaystyle\int \frac{\ln x}{x}\,dx$

(f) $\displaystyle\int \sin^3 x\cos x\,dx$

Integrals of the form $\frac{f'(x)}{f(x)}$

22.

(a) Find $\displaystyle\int \frac{3x^2+2}{x^3+2x}\,dx$.

(b) Find $\displaystyle\int \frac{e^x}{e^x+3}\,dx$.

(c) By writing $\tan x = \dfrac{\sin x}{\cos x}$, show that $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C$.

(d) Evaluate $\displaystyle\int_0^1 \frac{x+1}{x^2+2x+3}\,dx$, giving an exact answer.

Definite integrals by substitution

23.

(a) Evaluate $\displaystyle\int_0^2 x^2\sqrt{x^3+1}\,dx$ by converting the limits.

(b) Evaluate the same integral by substituting back to x first, and confirm that the answers agree.

(c) Evaluate $\displaystyle\int_0^{\pi/2} \sin^2 x\cos x\,dx$ by converting the limits.

Concept check

24. A student evaluates $\int_0^1 2x\left(x^2+1\right)^3 dx$ with $u = x^2+1$, and writes

$$\int_0^1 u^3\,du = \left[\frac{u^4}{4}\right]_0^1 = \frac{1}{4}.$$

Identify the error, and find the correct value.

15.6 Integration by Parts HL

Substitution reverses the chain rule; **integration by parts** reverses the product rule. It handles products that substitution cannot, such as xe^x or $\ln x$.

KEY · IB HL

$$\int u\,\frac{dv}{dx}\,dx = uv - \int v\,\frac{du}{dx}\,dx$$

This is the form printed in the formula booklet. It names the factor to be integrated dv/dx, which is accurate but slow to use. It is quicker to call the two factors u and v directly, with v the one you integrate. Replace dv/dx by v in the booklet formula: its v then becomes $\int v\,dx$, and du/dx is u'. The formula becomes the working form used in this book.

KEY HL

$$\int u\,v\,dx = u\int v\,dx \;-\; \int\Big(\int v\,dx\Big)\,u'\,dx$$

The first function times the integral of the second, minus the integral of that integral times the derivative of the first.

The first factor, u, is copied once and differentiated once. The second, v, is only ever integrated, and the same $\int v\,dx$ appears twice. Work it out once and write it in both places.

Read from left to right, the formula says: multiply u by the integral of v, then subtract the integral of that same $\int v\,dx$ times the derivative of u.

Everything now depends on which factor you call u. Choose the one that becomes simpler when differentiated, so that the second integral is easier than the one you started with. The order **ILATE** is a reliable guide: whichever type appears first in this list should be u.

- **I** **Inverse trigonometric**: $\arcsin x$, $\arctan x$
- **L** **Logarithmic**: $\ln x$
- **A** **Algebraic**: x, x^2, $3x-1$

T **Trigonometric**: $\sin x$, $\cos x$

E **Exponential**: e^x, 2^x

In $\int xe^x\,dx$ the factors are algebraic and exponential, so A comes before E and $u = x$. In $\int x \ln x\,dx$ they are logarithmic and algebraic, so L comes before A and $u = \ln x$. The order works because differentiating reduces the first three types to something simpler, while trigonometric and exponential functions repeat themselves however often you differentiate them.

TIP ILATE is a guide, not a theorem. It gives the right choice for most integrals in this course, but the test that always applies is the one behind it: does the second integral end up easier than the first?

A single function that has no standard integral, such as $\ln x$ or $\arctan x$, can still be integrated by parts. Write it as the function times 1: take u as the function and $v = 1$, so $\int v\,dx = x$.

Worked through. In $\int xe^x\,dx$, let $u = x$ (so $u' = 1$) and $v = e^x$ (so $\int v\,dx = e^x$):

$$\int xe^x\,dx = xe^x - \int e^x \cdot 1\,dx = xe^x - e^x + C = e^x(x-1) + C.$$

Choosing $u = x$ is what makes the new integral, $\int e^x\,dx$, trivial. The other choice, $u = e^x$, would give $\int \frac{x^2}{2}e^x\,dx$, which has a higher power of x.

EXAMPLE 15.8

Find (a) $\displaystyle\int (x+1)\sin x\,dx$, (b) $\displaystyle\int_0^{\pi/2} (x+1)\cos x\,dx$.

(a) The factors are algebraic and trigonometric, so ILATE gives $u = x+1$ and $v = \sin x$, hence $\int v\,dx = -\cos x$ and $u' = 1$:

$$\int (x+1)\sin x\,dx = -(x+1)\cos x - \int (-\cos x)\,dx = -(x+1)\cos x + \sin x + C.$$

Watch the two minus signs. Integrating $\sin x$ gives $-\cos x$, and the formula then subtracts the integral of that, which turns the sign back.

(b) Same choice, now with $v = \cos x$ and $\int v\,dx = \sin x$:

$$\int (x+1)\cos x\,dx = (x+1)\sin x - \int \sin x\,dx = (x+1)\sin x + \cos x + C.$$

Now apply the limits:

$$\int_0^{\pi/2} (x+1)\cos x\,dx = \Big[(x+1)\sin x + \cos x\Big]_0^{\pi/2} = \Big(\frac{\pi}{2} + 1 + 0\Big) - (0+1) = \frac{\pi}{2}.$$

For a definite integral by parts, find the whole antiderivative first, then substitute the limits once at the end.

EXAMPLE 15.9

Find $\displaystyle\int \ln x\, dx$.

There is only one function, but write $u = \ln x$ and $v = 1$, so $u' = \dfrac{1}{x}$ and $\int v\, dx = x$:

$$\int \ln x\, dx = x\ln x - \int x \cdot \frac{1}{x}\, dx = x\ln x - x + C.$$

The same choice, $v = 1$, integrates $\arctan x$.

15.6.1 Repeated integration by parts

If one application of parts leaves an integral that still needs parts, apply it again. Each round of differentiation reduces the power of x by one.

Worked through. To find $\displaystyle\int x^2 e^{-x}\, dx$, apply parts twice. First round, $u = x^2$ and $v = e^{-x}$, so $\int v\, dx = -e^{-x}$ and $u' = 2x$:

$$\int x^2 e^{-x}\, dx = -x^2 e^{-x} + 2\int x e^{-x}\, dx.$$

The remaining integral needs parts a second time, with $u = x$:

$$\int x e^{-x}\, dx = -x e^{-x} + \int e^{-x}\, dx = -e^{-x}(x+1) + C.$$

Substituting that back,

$$\int x^2 e^{-x}\, dx = -x^2 e^{-x} - 2e^{-x}(x+1) + C = -e^{-x}\left(x^2 + 2x + 2\right) + C.$$

A power x^n takes n rounds of parts to clear, since only the power changes from round to round; the exponential returns unchanged each time.

A different pattern appears when *neither* factor simplifies. For $e^x \sin x$, two rounds of parts bring the original integral back, and you solve for it like an equation.

EXAMPLE 15.10

Find $I = \displaystyle\int e^x \sin x\, dx$.

Parts with $u = \sin x$:

$$I = e^x \sin x - \int e^x \cos x\, dx.$$

Parts again on the new integral, $u = \cos x$:

$$\int e^x \cos x\, dx = e^x \cos x + \int e^x \sin x\, dx = e^x \cos x + I.$$

Substitute back:

$$I = e^x \sin x - e^x \cos x - I \implies 2I = e^x(\sin x - \cos x) \implies I = \tfrac{1}{2}e^x(\sin x - \cos x) + C.$$

The original integral reappeared on the right-hand side, which turned the problem into an equation for I. Watch for this whenever neither factor gets simpler: repeated rounds of parts often bring I back, and algebra gives the answer.

15.6.2 Choosing a method

Five methods are now available, and starting the wrong one costs time and marks. The choice is settled by looking at the integrand before writing anything.

What you see	Method	Example
A standard form exactly, or with a linear inside	read it off, dividing by the inner coefficient	$\cos 3x,\ e^{2x},\ (2x+1)^5$
A function together with a multiple of its own derivative	substitution	$2x(x^2+1)^3,\ \dfrac{2x}{x^2+1}$
A product of two unlike functions, one of which simplifies when differentiated	parts	$xe^x,\ x\sin x,\ \ln x$
A fraction whose denominator factorises	partial fractions, then logarithms	$\dfrac{1}{x^2+3x+2}$
A fraction whose denominator is a quadratic with no real roots	complete the square, then arctangent	$\dfrac{1}{x^2+2x+5}$

Work down the list in that order, because the earlier tests are quicker to apply and the earlier methods are shorter. Two habits help further. Check first whether the fraction is improper, since dividing may remove the problem entirely. And if a substitution leaves an integral no easier than the one you started with, the substitution was the wrong one; go back rather than pressing on.

YOUR TURN 15.6 Integration by Parts HL

One application of parts

25. Find each integral by parts.

(a) $\displaystyle\int x\cos x\,dx$

(b) $\displaystyle\int xe^{2x}\,dx$

(c) $\displaystyle\int x\ln x\,dx$

(d) $\displaystyle\int \arctan x\,dx$

(e) $\displaystyle\int x^2\ln x\,dx$

(f) $\displaystyle\int x\sec^2 x\,dx$ *(Hint: $\int \tan x\,dx = -\ln|\cos x| + C$, from Question 22)*

Definite integrals and the choice of u

26.

(a) Evaluate $\displaystyle\int_0^{\pi} x\sin x\,dx$.

(b) Evaluate $\displaystyle\int_0^{1} xe^{x}\,dx$.

(c) State which factor you chose as u in part (b), and why the other choice would not help.

(d) Use ILATE to say which factor should be u in each of $\displaystyle\int x^2e^x\,dx$, $\displaystyle\int x\arctan x\,dx$ and $\displaystyle\int e^x\sin x\,dx$.

(e) Explain why the third integral in part (d) behaves differently from the first two.

Applying parts twice

27. Find each integral. Each needs integration by parts twice.

(a) $\displaystyle\int x^2e^x\,dx$

(b) $\displaystyle\int x^2\sin x\,dx$

When the original integral returns

28. Find each integral. After two applications of parts the original integral appears again; solve for it.

(a) $\displaystyle\int e^x\cos x\,dx$

(b) $\displaystyle\int e^{2x}\sin x\,dx$

Concept check

29. A student writes $\int xe^x\,dx = \frac{x^2}{2}e^x + C$, integrating each factor separately. Show, by differentiating, that this is wrong, and find the correct integral.

15.7 Areas Between Curves

To find the area between two curves, integrate the difference: the upper curve minus the lower. The signed-area problem disappears, because the difference is positive wherever the first curve is on top, regardless of the axis.

KEY

$$A = \int_a^b (y_{\text{top}} - y_{\text{bottom}})\,dx$$

The limits a and b are usually the x-coordinates where the curves intersect, found by setting them equal. To decide which curve is on top, test one convenient point between the intersections. If the curves cross inside the interval, split the integral at each crossing.

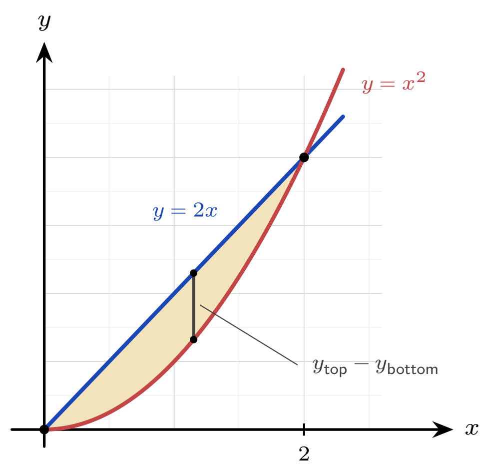

Figure 15.7 The region enclosed between $y = 2x$ and $y = x^2$. The curves meet where $2x = x^2$, at $x = 0$ and $x = 2$, and the line is the upper boundary in between, so the area is $\int_0^2 (2x - x^2)\,dx$.

Worked through. For the region in the figure, the curves meet where $2x = x^2$, that is at $x = 0$ and $x = 2$. On $(0, 2)$ the line $y = 2x$ is above, so

$$A = \int_0^2 (2x - x^2)\,dx = \left[x^2 - \tfrac{x^3}{3}\right]_0^2 = 4 - \tfrac{8}{3} = \tfrac{4}{3}.$$

Nothing in the method requires the curves to be polynomials. The steps are the same for trigonometric and exponential boundaries: find where they cross, decide which is on top, and integrate the difference.

EXAMPLE 15.11

Find the area enclosed between $y = \sin x$ and $y = \cos x$ between their first two intersections for $x \geq 0$.

They meet where $\sin x = \cos x$, that is $\tan x = 1$, so the first two intersections are at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

On $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ the sine is above: at $x = \frac{\pi}{2}$, $\sin\frac{\pi}{2} = 1$ while $\cos\frac{\pi}{2} = 0$. So

$$A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx = \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4}.$$

At the upper limit both $\cos x$ and $\sin x$ equal $-\frac{\sqrt{2}}{2}$, so the bracket is $\sqrt{2}$; at the lower limit both equal $\frac{\sqrt{2}}{2}$ and the bracket is $-\sqrt{2}$:

$$A = \sqrt{2} - \left(-\sqrt{2}\right) = 2\sqrt{2} \approx 2.83.$$

Testing the point $x = \frac{\pi}{2}$ decided which curve is on top, and so the sign of the integrand.

15.7.1 Area between a curve and the y-axis HL

Sometimes it is easier to integrate with respect to y: express x as a function of y and accumulate horizontal slices. This gives the area between the curve and the y-axis.

KEY · IB HL

$$A = \int_a^b |x|\,dy$$

Worked through. To find the area between the curve $x = y^2$ and the y-axis from $y = 0$ to $y = 2$, integrate with respect to y:

$$A = \int_0^2 y^2\,dy = \left[\tfrac{y^3}{3}\right]_0^2 = \tfrac{8}{3}.$$

In x the same area needs $\int_0^4 (2 - \sqrt{x})\,dx$: the line $y = 2$ minus the curve. Integrating in y is shorter.

YOUR TURN 15.7 Areas Between Curves

Finding an enclosed area

30. Consider the region enclosed by $y = x^2$ and $y = x + 2$.

(a) Find the x-coordinates where the two curves meet.
(b) State which curve is the upper boundary between those values.
(c) Find the area of the enclosed region.
(d) State what answer you would get by integrating $x^2 - (x + 2)$ instead, and explain what has gone wrong.

31. Find the area of the region enclosed by each pair of curves.

(a) $y = x$ and $y = x^2$.
(b) $y = 6 - x^2$ and $y = 2$.
(c) $y = x^2$ and $y = 2 - x^2$.
(d) $y = x^2$ and $y = 3x + 4$.
(e) $y = e^x$ and $y = e^{-x}$, between $x = 0$ and $x = 1$.

Curves that cross inside the interval

32.

(a) Find the total area between $y = x^2$ and $y = x$ from $x = 0$ to $x = 2$.
(b) The curves $y = x^3$ and $y = 4x$ enclose two regions. Find their total area.

Trigonometric boundaries

33.

(a) Find the area between $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{4}$.
(b) Find the total area between $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$.
(c) Find the area of the region enclosed by $y = \sin 2x$ and $y = \sin x$ for $0 \leq x \leq \frac{\pi}{3}$.

Integrating with respect to y

34.

(a) Find the area between the curve $x = y^2 + 1$ and the y-axis, from $y = 0$ to $y = 3$.
(b) Find the area between the curve $y = x^3$ and the y-axis, from $y = 1$ to $y = 8$.

Concept check

35. A student is finding the area enclosed by $y = x$ and $y = x^2 - 2$, and says: "*part of the region is below the x-axis, so I must split the integral where the curves cross the x-axis.*" Explain why no split is needed, and find the area.

15.8 Volumes of Revolution HL

Rotate the region under a curve through 360° about an axis and it sweeps out a solid. To find the volume, slice the solid into thin disks perpendicular to the axis. Each disk is a cylinder of radius y (the curve's height) and thickness dx, with volume $\pi y^2\,dx$. Summing the disks and letting their thickness shrink turns the sum into an integral.

KEY · IB HL

$$V = \int_a^b \pi y^2\,dx \quad \text{(about the } x\text{-axis)}, \qquad V = \int_a^b \pi x^2\,dy \quad \text{(about the } y\text{-axis)}$$

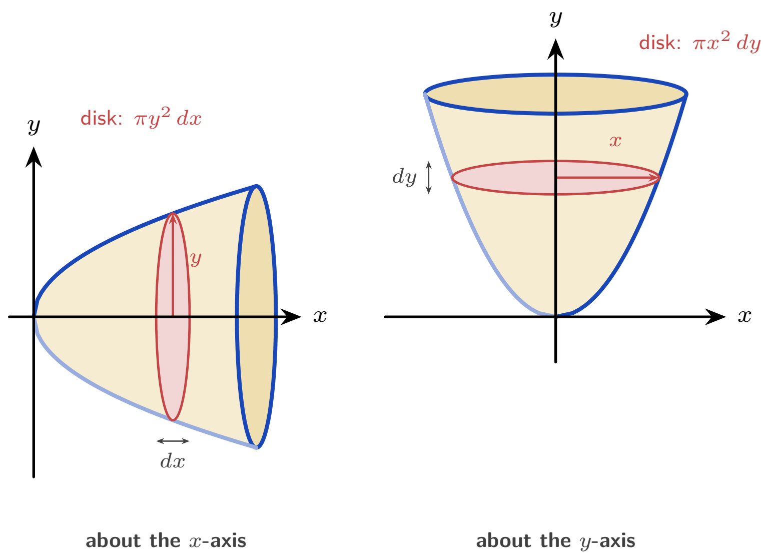

Figure 15.8 The same curve rotated about each axis. Turning it about the x-axis gives disks stacked along x: each has radius y and thickness dx, so its volume is $\pi y^2\,dx$. Turning it about the y-axis gives disks stacked along y: each has radius x and thickness dy, so its volume is $\pi x^2\,dy$. In both cases the integral adds the disks from one end of the solid to the other.

About the y-axis, each disk is swept by a horizontal strip running from the y-axis out to the curve, so the region rotated is the one between the curve and the y-axis, not the region under the curve. The two regions give different volumes in general. Write x^2 in terms of y and use y-values as the limits.

Worked through. Rotate the region under $y = x^3$, for $0 \le x \le 2$, through 360° about the x-axis. The volume is

$$V = \int_0^2 \pi (x^3)^2\,dx = \pi \int_0^2 x^6\,dx = \pi \left[\tfrac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}.$$

Square the function *before* integrating; squaring after is wrong.

EXAMPLE 15.12

The region between $y = x^2$ and the y-axis, from $y = 0$ to $y = 1$, is rotated about the y-axis. Find the volume.

About the y-axis, slice into disks of radius x and thickness dy, with $x^2 = y$:

$$V = \int_0^1 \pi x^2 \, dy = \pi \int_0^1 y \, dy = \pi \left[\tfrac{y^2}{2} \right]_0^1 = \frac{\pi}{2}.$$

The limits are now y-values, and x^2 is rewritten in terms of y.

The region rotated is the one between the curve and the y-axis, not the one under the curve.

Because the formula squares the function, a trigonometric curve produces $\sin^2$ or $\cos^2$, and those must be rewritten with a double angle before they can be integrated. See §15.4.

EXAMPLE 15.13

The region under $y = \sin 2x$ from $x = 0$ to $x = \frac{\pi}{4}$ is rotated $360°$ about the x-axis. Find the volume.

Apply the formula, squaring the function first:

$$V = \int_0^{\pi/4} \pi y^2 \, dx = \pi \int_0^{\pi/4} \sin^2 2x \, dx.$$

There is no standard integral for $\sin^2 2x$, so replace it using $\sin^2 2x = \dfrac{1 - \cos 4x}{2}$:

$$\pi \int_0^{\pi/4} \frac{1 - \cos 4x}{2} \, dx = \pi \left[\frac{x}{2} - \frac{\sin 4x}{8} \right]_0^{\pi/4} = \pi \left(\frac{\pi}{8} - 0 \right) = \frac{\pi^2}{8} \approx 1.23.$$

Two πs appear for different reasons, and they must be kept apart: one comes from the volume formula, the other from the upper limit.

EXAMPLE 15.14

The region under $y = e^x$ from $x = 0$ to $x = 2$ is rotated about the x-axis. Find the volume.

$$V = \pi \int_0^2 (e^x)^2 \, dx = \pi \int_0^2 e^{2x} \, dx = \pi \left[\tfrac{1}{2} e^{2x} \right]_0^2 = \frac{\pi}{2} \left(e^4 - 1 \right) \approx 84.2.$$

Squaring first is what turns e^x into e^{2x}, a linear composite. Squaring the integral instead of the function is a common error, and it gives a different answer entirely.

When the region lies between two curves, with y_{outer} further from the x-axis than y_{inner} across $a \leq x \leq b$, each slice is a disk with a disk-shaped hole in it, called a **washer**. Its volume is the volume of the outer disk minus that of the hole, $\pi y_{\text{outer}}^2\,dx - \pi y_{\text{inner}}^2\,dx$, so

$$V = \pi \int_a^b \left(y_{\text{outer}}^2 - y_{\text{inner}}^2\right) dx.$$

EXAMPLE 15.15

The region enclosed by $y = x$ and $y = x^2$ is rotated 360° about the x-axis. Find the volume. The curves meet at $x = 0$ and $x = 1$, and on $(0,1)$ the line is further from the axis, since $x > x^2$ there. So $y_{\text{outer}} = x$ and $y_{\text{inner}} = x^2$:

$$V = \pi \int_0^1 \left(x^2 - x^4\right) dx = \pi \left[\tfrac{x^3}{3} - \tfrac{x^5}{5}\right]_0^1 = \frac{2\pi}{15}.$$

CAUTION Square each function first, then subtract. Squaring the difference, $\pi \int_0^1 \left(x - x^2\right)^2 dx = \frac{\pi}{30}$, finds a different quantity and gives the wrong volume.

YOUR TURN 15.8 Volumes of Revolution HL

Rotating about the x-axis

36. Each region under the curve is rotated 360° about the x-axis. Find the volume.

(a) $y = x^2$, $0 \le x \le 2$.
(b) $y = \sqrt{x}$, $0 \le x \le 4$.
(c) $y = 2x$, $0 \le x \le 3$.
(d) $y = \dfrac{1}{x}$, $1 \le x \le 3$.
(e) $y = e^{-x}$, $0 \le x \le \ln 2$.
(f) $y = e^{3x}$, $0 \le x \le \frac{1}{3}$.

Squares of sine and cosine

37. Each region under the curve is rotated 360° about the x-axis. Find the exact volume.

(a) $y = \cos x$, $0 \le x \le \frac{\pi}{2}$.
(b) $y = \sin x$, $0 \le x \le \frac{\pi}{4}$.
(c) $y = 1 + \cos x$, $0 \le x \le \pi$.
(d) Explain why the volumes in this question need a double-angle rewriting, while those in Question 36(e) and (f) did not.

Rotating about the y-axis

38. Each region is rotated 360° about the y-axis. Find the volume.

(a) The region between $y = x^3$ and the y-axis, from $y = 0$ to $y = 8$.
(b) The region between $y = x^2 + 1$ and the y-axis, from $y = 1$ to $y = 3$.
(c) The region between $x = y^2$ and the y-axis, from $y = 0$ to $y = 2$.
(d) Your answer to part (c) equals your answer to Question 36(a). Explain why.

Washers

39. The region enclosed by each pair of curves is rotated 360° about the x-axis. Find the volume.

(a) $y = \sqrt{x}$ and $y = x^2$.
(b) $y = x + 2$ and $y = x^2$.
(c) $y = 4 - x^2$ and $y = 3$.

Concept check

40. For the region enclosed by $y = \sqrt{x}$ and $y = x^2$ in Question 39(a), a student writes $V = \pi \int_0^1 \left(\sqrt{x} - x^2\right)^2 dx$. Explain the error, and give the correct integral and volume.

15.9 Kinematics by Integration

In Ch. 14 you differentiated displacement to get velocity, and velocity to get acceleration. Integration reverses those two steps. Integrate acceleration to get velocity, and velocity to get displacement, using an initial condition each time to fix the constant.

EXAMPLE 15.16

A particle starts from rest at the origin, and its acceleration is $a = 6t - 4\,\text{m}\,\text{s}^{-2}$. Find its velocity and displacement at $t = 2$.

Integrate the acceleration, and use $v = 0$ at $t = 0$ to find the constant:

$$v = \int (6t - 4)\,dt = 3t^2 - 4t + C, \qquad 0 = 0 - 0 + C \implies C = 0.$$

Integrate the velocity, and use $s = 0$ at $t = 0$:

$$s = \int (3t^2 - 4t)\,dt = t^3 - 2t^2 + D, \qquad D = 0.$$

At $t = 2$: $v = 12 - 8 = 4\,\text{m}\,\text{s}^{-1}$ and $s = 8 - 8 = 0$ m. Each integration introduces one constant, so each needs one initial condition.

KEY · IB

$$\text{displacement} = \int_{t_1}^{t_2} v\,dt, \qquad \text{distance travelled} = \int_{t_1}^{t_2} |v|\,dt$$

Displacement is the signed change in position; distance is the total path length. They differ whenever the particle reverses direction, because backward motion subtracts from displacement but adds to distance. On a velocity–time graph the difference is visible: area below the axis counts as negative for displacement, and as positive for distance.

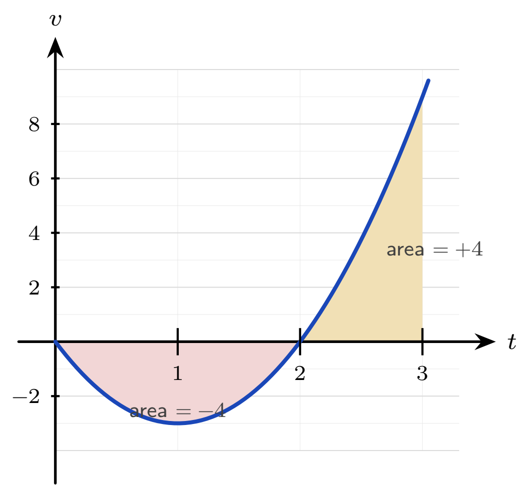

Figure 15.9 The velocity $v = 3t^2 - 6t$ over $0 \le t \le 3$. The particle moves backwards until $t = 2$ and forwards after it. Displacement adds the two signed areas, $-4 + 4 = 0$; distance adds their sizes, $4 + 4 = 8$ m.

Worked through. Take the velocity in the figure, $v(t) = 3t^2 - 6t\,\mathrm{m\,s^{-1}}$ over $0 \le t \le 3$. Displacement integrates v directly:

$$\int_0^3 (3t^2 - 6t)\,dt = \Big[t^3 - 3t^2\Big]_0^3 = 27 - 27 = 0.$$

The particle ends where it began. For distance, note $v = 3t(t-2)$ is negative on $(0, 2)$ and positive on $(2, 3)$, so split:

$$\left|\int_0^2 v\,dt\right| + \int_2^3 v\,dt = |-4| + 4 = 8\text{ m}.$$

Zero displacement but 8 m travelled: the particle went out 4 m and came back.

Oscillating motion is handled the same way, and it makes the difference between displacement and distance especially clear.

EXAMPLE 15.17

A particle has velocity $v(t) = 4\cos 2t\,\mathrm{m\,s^{-1}}$. Find its displacement and the distance travelled over $0 \le t \le \frac{\pi}{2}$ seconds.

Displacement integrates v directly. The antiderivative of $4\cos 2t$ is $2\sin 2t$:

$$\int_0^{\pi/2} 4\cos 2t\,dt = \Big[2\sin 2t\Big]_0^{\pi/2} = 2\sin\pi - 2\sin 0 = 0.$$

The particle finishes where it started. For the distance, first find where it turns: $v = 0$ when $\cos 2t = 0$, so $2t = \frac{\pi}{2}$ and $t = \frac{\pi}{4}$. Split the interval there:

$$\int_0^{\pi/4} 4\cos 2t\,dt = \Big[2\sin 2t\Big]_0^{\pi/4} = 2, \qquad \int_{\pi/4}^{\pi/2} 4\cos 2t\,dt = 0 - 2 = -2.$$

Distance adds the sizes: $|2| + |-2| = 4$ m. So the particle travels 2 m out and 2 m back, ending at its starting point.

Without a GDC the split is necessary: integrating v straight through, instead of $|v|$, returns 0, which is the displacement, not the distance. With a GDC, evaluate $\int_0^{\pi/2} |4\cos 2t|\,dt$ directly.

YOUR TURN 15.9 Kinematics by Integration

Displacement from velocity

41. A particle moves in a straight line, and t is the time in seconds. Find its displacement over the interval given.

(a) $v = 2t + 1$ m s^{-1}, $0 \le t \le 3$.
(b) $v = t^2 - 4$ m s^{-1}, $0 \le t \le 3$.
(c) $v = 3\cos t$ m s^{-1}, $0 \le t \le \frac{\pi}{2}$.
(d) $v = 6e^{-2t}$ m s^{-1}, $0 \le t \le 1$.

Integrating acceleration

42. A particle moves in a straight line, and t is the time in seconds.

(a) Its acceleration is $a = 6t$ m s^{-2} and $v(0) = 1$ m s^{-1}. Find $v(t)$.
(b) Its acceleration is $a = \sin t$ m s^{-2} and $v(0) = 0$. Find $v(t)$.
(c) Its acceleration is $a = 6t - 2$ m s^{-2}. At $t = 0$ its velocity is 3 m s^{-1} and its displacement from a fixed point O is 1 m. Find its displacement from O at $t = 2$.
(d) Its acceleration is $a = -4\sin 2t$ m s^{-2}. At $t = 0$ it is at O with velocity 2 m s^{-1}. Find its displacement s from O in terms of t.

Distance travelled

43. A particle moves in a straight line, and t is the time in seconds.

(a) Its velocity is $v = 4 - 2t$ m s^{-1}. Find the distance travelled over $0 \leq t \leq 4$.

(b) For the particle in part (a), find the displacement over the same interval, and explain why the two answers differ.

(c) Its velocity is $v = 3t^2 - 12$ m s^{-1}. Find the time at which it changes direction, and the distance travelled over $0 \leq t \leq 3$.

(d) Its velocity is $v = 1 - 2\sin t$ m s^{-1}. Find the time in $0 \leq t \leq \frac{\pi}{2}$ at which it changes direction, and the exact distance travelled over that interval.

Concept check

44. A particle moves with velocity $v = t^2 - 4t + 3$ m s^{-1}. A student finds the distance travelled over $0 \leq t \leq 3$ as $\left|\int_0^3 v\,dt\right|$. Explain why this is wrong, and find the distance travelled.

Chapter 15 · Integral Calculus

Reflections

TOK The proof at the end of this chapter establishes *that* the area under a curve and the reverse of a derivative are the same calculation. It does not explain *why* the world should be arranged that way. Two ideas invented for different purposes, one geometric and one algebraic, turned out to be one idea.

The integral now has two descriptions, the limit of a sum and the reverse of a derivative, and a theorem says they agree. Which one is the definition, and does the choice change what an integral is?

IM Long before calculus, Archimedes (3rd century BCE, Syracuse) found the area of a parabolic segment by his "method of exhaustion": filling the region with ever-smaller triangles and summing them. He effectively computed an integral two thousand years before Newton and Leibniz gave the operation a name and a reverse-differentiation shortcut. The idea of accumulation by slicing is older than the algebra we use to carry it out, and his triangles did the same job as the rectangles in §15.2.

RW Integration is summation made continuous, so it appears wherever a total is built from a varying rate. The total charge from a varying current, the work done by a changing force, the area of an irregular plot of land, the probability accumulated under a distribution curve (Ch. 12): each is an integral. Whenever a quantity is the running total of a rate, integration computes it.

A shape made by spinning a profile about an axis, such as a bottle, a lens or a turbine housing, is fixed completely by that one curve, so its volume is a single integral rather than a measurement. Engineers can size fuel tanks and machined parts this way, and the same integral gives the mass once the density is known.

EXT Not every function has an antiderivative in terms of familiar functions: $\int e^{-x^2}\,dx$, central to statistics, cannot be written with powers, exponentials, and trigonometric functions. Yet the definite integral still exists as a perfectly real area, and is computed numerically. The reach of "find the area" is wider than the reach of "reverse the derivative."

The Fundamental Theorem of Calculus

The Fundamental Theorem was stated in §15.2 and used throughout the chapter. Here it is proved. It is not examined, but it is the reason the chapter works. The result, reached by Barrow, Newton and Leibniz in the 17th century, states that if f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Assume first that $f(x) \geq 0$, so that every integral below is an area; the last step removes this assumption.

The **area function** $A(x) = \int_a^x f(t)\,dt$ is the area under the curve from a to a moving right end x. The letter t is only the variable of integration. The area of any band is then a difference:

$$\text{area from } x_1 \text{ to } x_2 \;=\; A(x_2) - A(x_1).$$

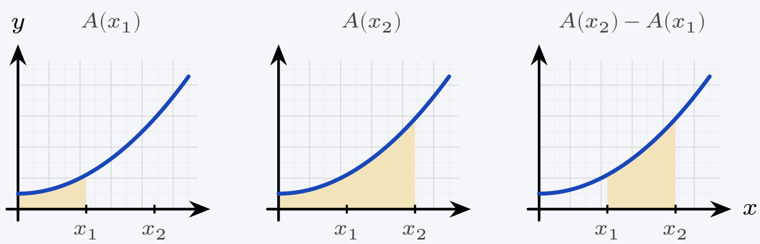

Figure 15.10 Shade up to x_2, remove what was shaded up to x_1, and the band between them remains.

So what kind of function is A? When x increases by a small amount h, the extra area $A(x + h) - A(x)$ is one thin strip of width h.

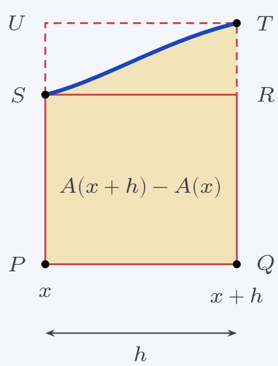

Figure 15.11 One strip, enlarged. Its area $A(x + h) - A(x)$ contains the solid rectangle $PQRS$ and fits inside the dashed rectangle $PQTU$, both of width h.

The short rectangle has height $f(x)$ and the tall one $f(x + h)$, so

$$f(x)\,h \;\leq\; A(x+h) - A(x) \;\leq\; f(x+h)\,h \qquad \Longrightarrow \qquad f(x) \;\leq\; \frac{A(x+h) - A(x)}{h} \;\leq\; f(x+h).$$

As $h \to 0$, $f(x + h) \to f(x)$ because f is continuous. The middle term, which is the definition of $A'(x)$, is squeezed between them:

$$A'(x) = f(x).$$

The same squeeze works for $h < 0$. So the area function is **an antiderivative of** f. If f falls across the strip the rectangles swap roles, and if it rises and falls, use its least and greatest values on the strip.

Now let F be any antiderivative of f. Then $(A - F)' = 0$ on the interval, so $A = F + C$ for a constant C. The area from a to a is zero, so $C = -F(a)$, and at $x = b$

$$\int_a^b f(x)\,dx \;=\; A(b) \;=\; F(b) - F(a).$$

If f takes negative values, apply the argument to $f + k$, with k chosen so that $f + k \geq 0$ on $[a, b]$. Its antiderivative is $F + kx$, and the extra rectangle area $k(b - a)$ appears on both sides and cancels. ■

The proof also shows that

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Accumulating f and then differentiating returns f. This is often called the first part of the Fundamental Theorem. It is not examined.

End-of-Chapter Problems
Chapter 15: Integral Calculus

1. A curve has $\dfrac{dy}{dx} = 3x^2 - 4x + 1$ and passes through $(1, 2)$.

(a) Find the equation of the curve. [3]

(b) Find the value of y when $x = 2$. [2]

2. Find the following integrals.

(a) $\displaystyle\int 6e^{1-2x}\,dx$ [2]

(b) $\displaystyle\int (4x-1)^6\,dx$ [2]

(c) $\displaystyle\int 3\sin\frac{x}{2}\,dx$ [2]

3. Evaluate $\displaystyle\int_1^4 \left(\frac{2}{\sqrt{x}} + 1\right) dx$. [4]

4. Consider $y = x^2 - 4$.

(a) Find the x-intercepts. [2]

(b) Find the area enclosed between the curve and the x-axis. [4]

5. The curves $y = x^2$ and $y = 8 - x^2$ intersect at two points.

(a) Find the coordinates of the intersection points. [2]

(b) Find the area enclosed between the curves. [4]

6.

(a) Using the substitution $u = x^2 + 4$, find $\displaystyle\int \frac{x}{x^2+4}\,dx$. [3]

(b) Hence evaluate $\displaystyle\int_0^2 \frac{x}{x^2+4}\,dx$, giving an exact answer. [2]

7.

(a) Find $\displaystyle\int x\sin 3x\,dx$ using integration by parts. [3]

(b) Evaluate $\displaystyle\int_0^{\pi/3} x\sin 3x\,dx$, giving an exact answer. [2]

8. A particle starts at the origin with velocity $2\,\mathrm{m\,s^{-1}}$ and has acceleration $a = 6t - 8\,\mathrm{m\,s^{-2}}$.

(a) Find the velocity $v(t)$. [2]
(b) Find the displacement $s(t)$. [2]
(c) Find the displacement when $t = 3$, and interpret its sign. [2]

9. Consider the region enclosed by $y = x^2$, the y-axis, and $y = 4$.

(a) Find the area of the region. [3]
(b) Find the volume when this region is rotated about the y-axis. [3]

10.

(a) Find $\displaystyle\int \frac{1}{9x^2+4}\,dx$. [3]
(b) Evaluate $\displaystyle\int_0^{2/3} \frac{1}{9x^2+4}\,dx$, giving an exact answer. [2]

11. The velocity of a particle is $v(t) = 3t^2 - 12t + 9\,\mathrm{m\,s^{-1}}$, $t \geq 0$.

(a) Find the times when the particle is at rest. [2]
(b) Find the total distance travelled in the first 3 seconds. [4]

12. Consider the curve $y = x^2 - 5x + 4$ on the interval $0 \leq x \leq 2$.

(a) Find where the curve crosses the x-axis in this interval. [1]
(b) Find the total area enclosed between the curve and the x-axis on the interval. [4]

13. Find the area of the region enclosed by the curve $y = 4x - x^2$ and the line $y = x$. [4]

14. Using the substitution $u = x^2 + 1$, evaluate $\displaystyle\int_0^2 x(x^2+1)^3\,dx$. [4]

15. Evaluate $\displaystyle\int_0^2 \frac{x^2}{x^3+1}\,dx$, giving your answer in the form $a\ln 3$, where $a \in \mathbb{Q}$. [4]

16. Find $\displaystyle\int \frac{x+7}{(x-1)(x+3)}\,dx$. [5]

17. Find $\displaystyle\int \frac{1}{x^2+8x+20}\,dx$. [4]

18.

(a) Using the substitution $u = \sqrt{x}$, show that $\displaystyle\int_0^4 \frac{1}{1+\sqrt{x}}\,dx = 4 - 2\ln 3$. [6]

(b) Show that $4x - x^2 = 4 - (x-2)^2$. Hence evaluate $\displaystyle\int_1^2 \frac{1}{\sqrt{4x - x^2}}\,dx$, giving an exact answer. [5]

19. Find the following integrals.

(a) $\displaystyle\int \ln x\,dx$, by writing $\ln x = 1 \times \ln x$ [3]

(b) $\displaystyle\int \sqrt{x}\,\ln x\,dx$ [4]

20.

(a) Use integration by parts twice to find $\displaystyle\int x^2 \cos 2x\,dx$. [5]

(b) Hence evaluate $\displaystyle\int_0^{\pi/4} x^2 \cos 2x\,dx$, giving an exact answer. [2]

21. The region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$ is rotated about the x-axis.

(a) Find the volume generated. [3]

(b) The same curve, written $x = y^2$ for $0 \le y \le 2$, together with the y-axis, bounds a region. This region is rotated about the y-axis. Find the volume generated. [4]

22. The area under $y = x^2 + 1$ from $x = 0$ to $x = 3$ is to be estimated by three rectangles of width 1.

(a) Find the estimate obtained by taking the height of each rectangle at the left of its strip. [3]

(b) Find the estimate obtained by taking the height at the right. [2]

(c) Evaluate $\displaystyle\int_0^3 (x^2 + 1)\,dx$ and verify that it lies between the two estimates. [3]

(d) State, with a reason, whether the left estimate would increase or decrease if six rectangles were used instead. [2]

23. The curve $y = x^2$ and the line $y = 2 - x$ enclose a region R.

(a) Find the x-coordinates of the two points of intersection. [3]

(b) Sketch the curve and the line, and shade R. [2]

(c) Show that the area of R is $\dfrac{9}{2}$. [5]

(d) Explain why integrating $y_{\text{curve}} - y_{\text{line}}$ instead would give the wrong sign. [2]

24. This question uses integration by parts throughout.

(a) Find $\displaystyle\int \frac{\ln x}{x^2}\,dx$. [4]

(b) Hence evaluate $\displaystyle\int_1^e \frac{\ln x}{x^2}\,dx$, giving your answer in terms of e. [3]

(c) Evaluate $\displaystyle\int_0^\pi x\sin\frac{x}{2}\,dx$. [5]

(d) In part (c), state which factor you took as u, and explain what would go wrong with the other choice. [2]

25. Consider the curve $y = x^3$ for $0 \le x \le 1$.

(a) The region between the curve and the x-axis is rotated 360° about the x-axis. Show that the volume is $\frac{\pi}{7}$. [4]

(b) The region between the curve and the y-axis is rotated 360° about the y-axis. Find this volume, giving an exact answer. [5]

(c) State which of the two solids has the greater volume, and explain the answer by describing the two regions. [3]

26. A particle moves in a straight line with acceleration $a = 6 - 2t\ \mathrm{m\,s^{-2}}$, and is at rest at $t = 0$.

(a) Show that $v = 6t - t^2$. [3]

(b) Find the time at which the particle is next at rest. [2]

(c) Find the maximum velocity, and the time at which it occurs. [3]

(d) Find the distance travelled from $t = 0$ until the particle is next at rest. [4]

(e) Explain why, for this motion, the distance travelled equals the displacement. [2]

27. Name the method for each integral before using it.

(a) $\displaystyle\int \frac{x^2+3}{x-1}\,dx$ [4]

(b) $\displaystyle\int_1^2 \frac{2x-1}{x^2-x+1}\,dx$ [4]

(c) $\displaystyle\int_0^{\pi/2} \sin x\cos^2 x\,dx$ [4]

(d) $\displaystyle\int \frac{1}{x^2-4x+13}\,dx$ [4]

28. GDC The curves $y = 3\cos x$ and $y = x^2$ enclose a region R.

(a) Find the x-coordinates of the two points of intersection, correct to 3 s.f. [2]

(b) Find the area of R. [3]

(c) R is rotated 360° about the x-axis. Find the volume of the solid generated. [3]

29. GDC A particle moves in a straight line with velocity $v(t) = 5\sin\left(\frac{t^2}{4}\right)\ \mathrm{m\,s^{-1}}$ for

$0 \le t \le 4$ seconds.

(a) Find the time $t > 0$ at which the particle changes direction. [2]
(b) Find the acceleration of the particle when $t = 3$. [2]
(c) Find the displacement of the particle over $0 \le t \le 4$. [2]
(d) Find the total distance travelled over $0 \le t \le 4$. [3]

30. The region R is enclosed by $y = \sin x$, $y = \cos x$ and the y-axis.

(a) Show that the two curves first meet at $x = \frac{\pi}{4}$. [2]
(b) Find the area of R. [4]
(c) R is rotated 360° about the x-axis. Show that the volume is $\frac{\pi}{2}$. [5]

31. A particle moves in a straight line with velocity $v(t) = 6\sin 2t\,\mathrm{m\,s^{-1}}$, for $0 \le t \le \pi$ seconds.

(a) Find the times at which the particle is at rest. [3]
(b) Find the displacement over $0 \le t \le \pi$. [3]
(c) Find the total distance travelled over the same interval. [4]
(d) Explain why the two answers differ. [2]

32. A particle starts from rest at the origin with acceleration $a(t) = 4e^{-2t}\,\mathrm{m\,s^{-2}}$.

(a) Show that $v(t) = 2 - 2e^{-2t}$. [3]
(b) Find the displacement over $0 \le t \le 1$. [4]
(c) State the velocity the particle approaches as t grows large, and explain what is happening physically. [2]

33. Consider the region enclosed by $y = e^x$, $y = e^{-x}$ and the line $x = 1$.

(a) Show that the two curves meet at $x = 0$. [1]
(b) Find the area of the region. [4]
(c) The region is rotated 360° about the x-axis. Find the volume, giving your answer in terms of e and π. [5]

34. GDC The inside of a vase is modelled by rotating the curve $y = 3 + \sin(0.5x)$, $0 \le x \le 12$, through 360° about the x-axis, where x and y are measured in centimetres. The vase stands on its base, at $x = 0$, with the x-axis vertical.

(a) Write down an integral for the volume of the vase. [1]
(b) Find the volume of the vase, in cm^3. [2]
(c) Water is poured into the empty vase until it is exactly half full. Find the depth of the water. [4]
(d) Find the maximum internal radius of the vase, and the height x at which it occurs. [2]

Challenge Questions

35. Solids you already know. Several volume formulas from Ch. 7 can be recovered by rotating a curve about the x-axis. This question derives three of them, and then finds a single formula that contains two of them as special cases.

(a) The line $y = r$ for $0 \le x \le h$ is rotated about the x-axis. Show that the volume is $\pi r^2 h$, and name the solid. [3]

(b) The line $y = \frac{r}{h}x$ for $0 \le x \le h$ is rotated about the x-axis. Show that the volume is $\frac{1}{3}\pi r^2 h$. [4]

(c) The semicircle $y = \sqrt{r^2 - x^2}$ for $-r \le x \le r$ is rotated about the x-axis. Show that the volume is $\frac{4}{3}\pi r^3$. [4]

(d) Let $m > 0$. The curve $y = r\left(\frac{x}{h}\right)^m$ for $0 \le x \le h$ is rotated about the x-axis. Show that the volume is

$$V = \frac{\pi r^2 h}{2m+1}.$$

[4]

(e) Show that the formula in part (d) also gives the results of parts (a) and (b), taking $m = 0$ for the line $y = r$. Use it to explain where the factor $\frac{1}{3}$ in the volume of the cone comes from. [3]

(f) Every solid in part (d) fits inside the cylinder of part (a). Find the value of m for which the solid has exactly half the volume of that cylinder, and describe the curve that is rotated. [2]

(g) Many containers, such as a bowl, a vase or a glass, have a profile close to one of the curves in part (d). Choose one, and describe how you would find a value of m that models it and test the model against a measured volume. [2]

36. Damped oscillation. A pendulum swinging in air moves in a way that the graph of $y = e^{-x}\sin x$ describes: it oscillates, and each swing is smaller than the last. This question finds the area under such curves by two methods and then adds up the areas of all the swings. Throughout, a and b are non-zero real constants, and

$$I = \int e^{ax}\cos bx\,dx, \qquad J = \int e^{ax}\sin bx\,dx.$$

(a) Apply integration by parts once to each integral, differentiating e^{ax} each time, to show that, apart from constants of integration,

$$bI = e^{ax}\sin bx - aJ \qquad \text{and} \qquad bJ = -e^{ax}\cos bx + aI.$$

[4]

(b) Solve these two equations simultaneously to show that

$$I = \frac{e^{ax}\left(a\cos bx + b\sin bx\right)}{a^2+b^2} + C,$$

and find J in the same form. [4]

(c) A second route uses complex numbers (Ch. 17). You are given that

$$e^{(a+ib)x} = e^{ax}\left(\cos bx + i\sin bx\right) \qquad \text{and} \qquad \int e^{cx}\,dx = \frac{e^{cx}}{c} + C,$$

where the second result holds when c is a complex constant. Deduce that $I + iJ = \dfrac{e^{(a+ib)x}}{a+ib} + C$. By multiplying the numerator and denominator by $a - ib$, recover both results of part (b) at once. [4]

(d) Let $f(x) = e^{-x}\sin x$ and, for $k = 0, 1, 2, \ldots$, let $A_k = \displaystyle\int_{k\pi}^{(k+1)\pi} f(x)\,dx$, the signed area of the kth swing. Show that $A_0 = \frac{1}{2}\left(1 + e^{-\pi}\right)$, and find A_1 and A_2 exactly. [3]

(e) Conjecture a relationship between A_{k+1} and A_k. Prove it for every $k \geq 0$ by using the substitution $x = u + \pi$ in A_{k+1}. [4]

(f) Hence show that $\displaystyle\int_0^{\infty} e^{-x}\sin x\,dx = \frac{1}{2}$, and confirm this value using your expression for J from part (b). [3]

(g) For the curve $y = e^{-cx}\sin x$ with $c > 0$, investigate how the ratio of the areas of successive swings depends on c. Suggest how timing and measuring the successive swings of a real pendulum could be used to estimate its value of c. [2]

37. A reduction formula. Let $I_n = \displaystyle\int_0^{\pi/2} \sin^n x\,dx$ for $n \geq 0$.

(a) Evaluate I_0 and I_1 directly. [3]

(b) Writing $\sin^n x = \sin^{n-1} x \cdot \sin x$ and using integration by parts, show that

$$I_n = \frac{n-1}{n} I_{n-2}, \qquad n \geq 2.$$

[6]

(c) Use the formula to find I_2, I_3, I_4 and I_5 exactly. [4]

(d) Explain why I_n involves π when n is even but not when n is odd. [3]

(e) Explain why $I_{n+1} \leq I_n$ for every $n \geq 0$. Investigate what happens to the ratio $\dfrac{I_{2m}}{I_{2m+1}}$ as m increases, and what this suggests about writing $\frac{\pi}{2}$ as an infinite product of fractions. [3]

38. Exact areas from rectangles. The notes estimate the area under $y = x^2$ on $[0, 1]$ with rectangles, and then evaluate it exactly with the Fundamental Theorem of Calculus. This question finds the same exact value from the rectangles alone. Divide $[0, 1]$ into n strips of equal width $\frac{1}{n}$, and let U_n and L_n be the sums of the rectangles whose heights are read at the right and at the left of each strip.

(a) Show that $U_n = \frac{1}{n^3}\sum_{r=1}^{n} r^2$, and write L_n as a sum in the same way. [3]

(b) Given that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$, show that $U_n = \frac{(n+1)(2n+1)}{6n^2}$, and find L_n in the same form. Evaluate L_{10} and U_{10}. [4]

(c) Show that $U_n - L_n = \frac{1}{n}$. Hence, without using the Fundamental Theorem, show that the area under $y = x^2$ on $[0, 1]$ is exactly $\frac{1}{3}$. [3]

(d) Given that $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$, use the same method to show that the area under $y = x^3$ from $x = 0$ to $x = b$, where $b > 0$, is $\frac{1}{4}b^4$. [4]

(e) Conjecture the area under $y = x^p$ on $[0, 1]$ for a positive integer p. State what the leading term of a formula for $\sum_{r=1}^{n} r^p$ would have to be for the method to give your conjecture. [3]

(f) The method depends on having a formula for the sum. Investigate a curve for which a different kind of sum can be evaluated exactly, for example $y = 2^x$ on $[0, 1]$, whose rectangle sum is a geometric series (Ch. 2). [2]