CALCULUS

Chapter 14

Derivative Applications

14

SYLLABUS 5.2 5.7 5.8 5.9 5.14

Throw a ball straight up. It rises, it slows, and at the very top it stops for an instant before it falls back. At that instant its velocity is exactly zero. A moment earlier the ball was moving up; a moment later it is moving down. The top of the flight is the one place where it is doing neither.

That instant points to a method for a large class of problems. The top of the flight is where the height is greatest, and it is also where the velocity is zero. The same pattern holds widely: for a smooth quantity at an interior point, when it reaches its largest or smallest value its rate of change is zero at that moment, and the curve is flat.

In this chapter the derivative stops being something you only calculate and becomes something you use. By the end of this chapter you will be able to:

- find the turning points of a curve, and decide which is a maximum and which a minimum;
- read the shape of a graph from the signs of f' and f'';
- find the largest or smallest value of a quantity under a constraint;
- describe motion along a straight line;
- find the gradient of a curve that is not written as $y = f(x)$ (implicit differentiation), and link the rates of quantities that change together (related rates).

A derivative is not only a gradient: it is a rate of change, and many real questions ask how fast something changes or where a quantity is greatest or least.

14.1 Stationary Points and Their Nature

When a quantity reaches its largest or smallest value, it has stopped increasing and has not yet started decreasing, or the reverse. The same condition, $f'(x) = 0$, marks that moment in many subjects. A projectile is at its greatest height when its vertical velocity is zero (§14.4). A profit is largest where its rate of change with respect to the number of items made is zero (§14.3). In physics a system is in equilibrium where its potential energy is stationary. In statistics the mode of a continuous distribution is at a maximum of its density function Ch. 12. To use the condition, first read what the sign of f' says about a curve.

14.1.1 Increasing and decreasing

The derivative gives the gradient, so its sign shows which way the curve is going. Where $f'(x) > 0$ the tangent rises from left to right, and a larger x gives a larger $f(x)$: the function is **increasing**. Where $f'(x) < 0$ the function is **decreasing**. Where $f'(x) = 0$ the tangent is horizontal.

> KEY If $f'(x) > 0$ for every x in an interval, then f is **increasing** on that interval.
> If $f'(x) < 0$ for every x in an interval, then f is **decreasing** on that interval.

To find where f is increasing, differentiate, factorise $f'(x)$, and solve the inequality $f'(x) > 0$. Solve $f'(x) < 0$ for where it is decreasing.

Worked through. To find the intervals on which $f(x) = x^3 - 9x^2$ is increasing, and those on which it is decreasing, differentiate and factorise:

$$f'(x) = 3x^2 - 18x = 3x(x - 6).$$

The product is zero at $x = 0$ and $x = 6$. It is positive when both factors have the same sign, so $f'(x) > 0$ for $x < 0$ and for $x > 6$. Between the roots the factors have opposite signs, so $f'(x) < 0$ for $0 < x < 6$.

So f is increasing for $x < 0$ and for $x > 6$, and decreasing for $0 < x < 6$. The behaviour can change only where $f'(x) = 0$, and those points are the subject of the rest of this section.

14.1.2 Stationary points

A **stationary point** is a point where the gradient is zero: the tangent is horizontal. Since the derivative gives the gradient, stationary points are exactly the solutions of $f'(x) = 0$. They are the points where a curve stops increasing or stops decreasing.

For a smooth curve, every local maximum and minimum is a stationary point, so the condition $f'(x) = 0$ finds them all. It does not find an extreme value at a corner, where the derivative does not exist: $f(x) = |x|$ has a minimum at $x = 0$ and no stationary point there.

See Ch. 13.

DEF A **stationary point** of f occurs where $f'(x) = 0$. At a **local maximum** the curve changes from increasing to decreasing; at a **local minimum**, from decreasing to increasing.

The word local means highest or lowest among nearby points only. A local maximum need not be the largest value of the function: the curve below has a local maximum at $(-1, 2)$ and still increases without limit to the right.

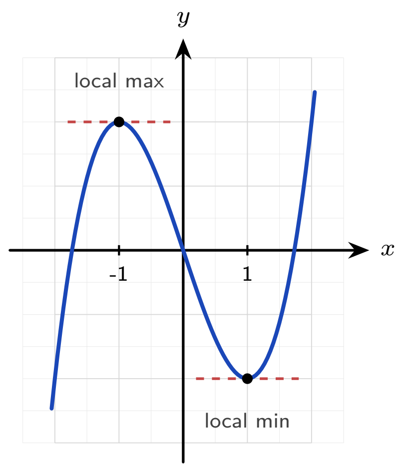


Figure 14.1 At each stationary point the tangent is horizontal. The curve changes from increasing to decreasing at the maximum, and from decreasing to increasing at the minimum.

A stationary point is a candidate, not an answer. A zero gradient says only that the curve is flat there. Flat can mean a local maximum, a local minimum, or neither: $y = x^3$ is flat at the origin and keeps increasing through it. Two tests decide which case you have.

14.1.3 The first derivative test

The sign of f' just to the left and just to the right of a stationary point shows what the curve does there. Just before a maximum the curve rises, so $f' > 0$; just after it, the curve falls, so $f' < 0$. A minimum is the reverse. If f' has the same sign on both sides, the curve keeps rising, or keeps falling, through the point. It is then a **stationary point of inflexion** (§14.2).

Sign of f'	Shape of the curve	Nature of the point
$+$ then $-$	rising, then falling	local maximum
$-$ then $+$	falling, then rising	local minimum
same sign	rising throughout, or falling throughout	stationary point of inflexion

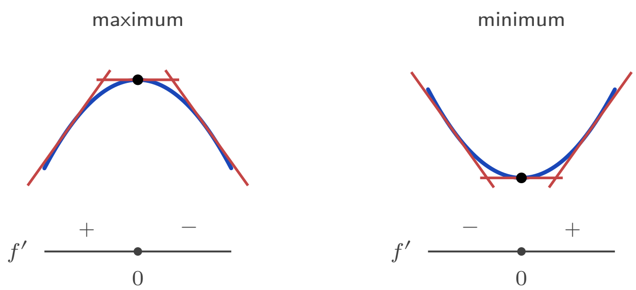


Figure 14.2 The first derivative test. The scarlet strokes are tangents: rising, flat, then falling at a maximum, and the reverse at a minimum. The strip below records the sign of f' on each side.

The method has three steps.

1. Solve $f'(x) = 0$ to find the stationary points.
2. Find the sign of f' just to the left and just to the right of each one. If $f'(x)$ is factorised, the sign of each factor gives the sign of f' without substituting any numbers.
3. Read the nature of each point from the table.

Worked through. To find and classify the stationary points of $f(x) = 3x^4 - 4x^3$, differentiate and factorise:

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1).$$

So $f'(x) = 0$ at $x = 0$ and $x = 1$.

The factor $12x^2$ is positive on both sides of 0 and on both sides of 1. So the sign of f' is the sign of $x - 1$: negative for $x < 1$, positive for $x > 1$.

At $x = 0$ the sign of f' is negative, then negative again. The curve falls through the point, so $(0, 0)$ is a stationary point of inflexion.

At $x = 1$ the sign changes from negative to positive, so the point is a local minimum. Since $f(1) = 3 - 4 = -1$, it is $(1, -1)$.

A repeated factor such as x^2 does not change sign, so it cannot produce a maximum or a minimum on its own.

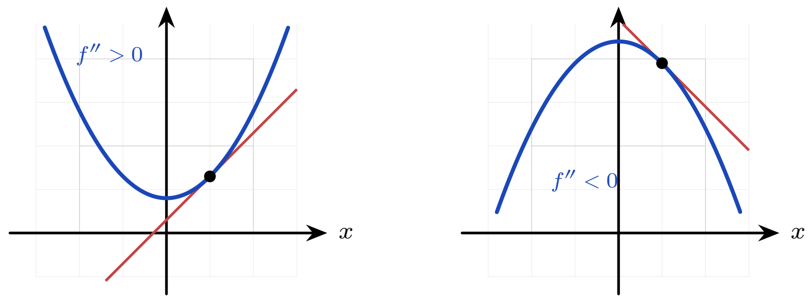
14.1.4 Concavity

The first derivative says whether a curve is going up or down. It does not say how the curve bends. Two curves can both rise from the same point, one more and more steeply, the other less and less steeply. The second derivative separates them, because f'' is the rate of change of the gradient §13.7.

Where $f''(x) > 0$ the gradient is increasing, and the curve is **concave up**: it bends upward, like a cup that holds water. Where $f''(x) < 0$ the gradient is decreasing, and the curve is **concave down**: it bends downward, like a cup turned upside down.

DEF The curve $y = f(x)$ is **concave up** on an interval where $f''(x) > 0$, and **concave down** on an interval where $f''(x) < 0$.

The tangent line gives a picture of the same fact. Where a curve is concave up it lies *above* its tangents, so a tangent touches the curve and the curve then rises away from it. Where the curve is concave down, it lies below its tangents.



concave up: curve *above* its tangent   concave down: curve *below* its tangent

Figure 14.3 The tangent picture of concavity. A tangent (scarlet) touches the curve at one point. Where the curve is concave up it stays above every tangent; where it is concave down it stays below.

Notice what f'' reports on: the gradient, not the height. Concave up means the gradient is increasing, and that can happen while the curve falls, because a gradient that goes from -5 to -1 has increased. So the signs of f' and f'' answer two separate questions.

Sign	f': going up or down?	f'': bending which way?
positive	**increasing**	**concave up** (bends upward)
negative	**decreasing**	**concave down** (bends downward)
zero	**stationary point**: the curve is flat there	**point of inflexion**, *if* the concavity also changes

The two signs are independent. A curve can be increasing and concave down at the same time: it rises by less and less. It can also be decreasing and concave up: it falls by less and less.

14.1.5 The second derivative test

At a stationary point the tangent is horizontal. A curve that is concave up there lies above its horizontal tangent, so the point is a local minimum. A curve that is concave down lies below it, so the point is a local maximum. One substitution into f'' therefore decides the nature of the point. This is usually faster than the first derivative test.

KEY At a stationary point where $f'(x) = 0$:

$$f''(x) > 0 \Rightarrow \text{local minimum}, \qquad f''(x) < 0 \Rightarrow \text{local maximum}.$$

If $f''(x) = 0$ this test gives no answer. Use the first derivative test instead.

The case $f''(x) = 0$ is a real gap in the test, and it does not mean the point is an inflexion. Both $y = x^4$ and $y = x^3$ have $f'(0) = 0$ and $f''(0) = 0$. Yet x^4 has a minimum at the origin, and x^3 has a stationary point of inflexion there. Only the sign of f' on each side separates them: for x^4 it is negative then positive, while for x^3 it is positive on both sides.

Worked through. To find and classify the stationary points of $f(x) = x^3 + 3x^2 - 9x + 2$, differentiate and solve $f'(x) = 0$:

$$f'(x) = 3x^2 + 6x - 9 = 3(x+3)(x-1) = 0 \implies x = -3 \text{ or } x = 1.$$

The second derivative is $f''(x) = 6x + 6$. At $x = -3$, $f''(-3) = -12 < 0$, so the point is a local maximum; $f(-3) = 29$, so it is $(-3, 29)$. At $x = 1$, $f''(1) = 12 > 0$, so the point is a local minimum; $f(1) = -3$, so it is $(1, -3)$. This cubic increases without limit to the right, so 29 is not the largest value of f.

The method does not depend on the function being a polynomial. Any function you can differentiate twice can be tested in the same way. State the domain whenever the function restricts it, as $\ln x$ requires $x > 0$. If $f''(x) > 0$ across the whole domain, the curve is concave up everywhere, so a stationary point there is the lowest point of the entire curve and not only a local minimum.

EXAMPLE 14.1

Find and classify the stationary point of $f(x) = x \ln x$, for $x > 0$.

The product rule gives

$$f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1.$$

Set $f'(x) = 0$: $\ln x = -1$, so $x = e^{-1} \approx 0.368$.

The second derivative is $f''(x) = \dfrac{1}{x}$, which is positive for every x in the domain. So the point is a local minimum, and

$$f\left(\tfrac{1}{e}\right) = \tfrac{1}{e} \ln \tfrac{1}{e} = -\tfrac{1}{e} \approx -0.368.$$

The minimum is at $\left(\frac{1}{e}, -\frac{1}{e}\right)$. The domain $x > 0$ comes from the logarithm, and since $f'' > 0$ on all of it, this is the lowest point of the entire curve.

YOUR TURN 14.1 Stationary Points and Their Nature

Increasing and decreasing

1. Find the intervals on which each function is increasing, and those on which it is decreasing.

(a) $f(x) = x^2 - 4x$
(b) $f(x) = x^4 - 2x^2$
(c) $f(x) = \sin x$, for $0 \le x \le 2\pi$
(d) $f(x) = xe^{-x}$

Concavity

2. Find the intervals on which each curve is concave up, and those on which it is concave down.

(a) $y = x^3$
(b) $y = x^3 - 6x^2$
(c) $y = \cos x$, for $0 \le x \le 2\pi$
(d) $y = \ln x$, for $x > 0$

Finding and classifying stationary points

3. Find the stationary points of each curve.

(a) $y = x^2 - 6x + 5$
(b) $y = x^3 - 12x$
(c) $y = 2x^3 - 3x^2 - 12x + 1$
(d) $y = x^4 - 2x^2$

4.

(a) Use the second derivative to classify the stationary point of $y = x^2 - 6x + 5$.
(b) Use the second derivative to classify both stationary points of $y = x^3 - 12x$.
(c) Find the stationary point of $y = x + \dfrac{1}{x}$ for $x > 0$, and state its nature.
(d) A curve has $\dfrac{dy}{dx} = (x-1)(x-4)$. Write down the x-coordinates of its stationary points, and state the nature of each without differentiating again.

5. Find the stationary point of each curve and determine its nature.

(a) $y = e^x - 3x$
(b) $y = \dfrac{\ln x}{x}$, for $x > 0$
(c) $y = \sin x + \cos x$, for $0 \le x \le 2\pi$ (there are two)
(d) $y = xe^{-2x}$

The first derivative test

6. The curve $y = x^3 - 3x^2 + 3x$ has one stationary point.

(a) Show that $x = 1$ is a stationary point.

(b) Explain why the second derivative test cannot classify it.

(c) Use the sign of $\frac{dy}{dx}$ on either side to classify it.

7.

(a) A curve has $\frac{dy}{dx} = (x+1)^2(x-2)$. Find the x-coordinates of its stationary points, and use the first derivative test to classify each one.

(b) Find and classify the stationary points of $f(x) = x^4 - 4x^3$.

Local and global values

8. Let $f(x) = 2x^3 - 3x^2 - 12x$ for $-2 \le x \le 4$.

(a) Find the local maximum and the local minimum of f.

(b) Find the greatest and least values of f on the interval, and explain why the local maximum is not the greatest value.

Concept check

9. A student writes: "*$f'(a) = 0$, so f has a maximum or a minimum at $x = a$.*" Give a counter-example, and state what must be checked before this conclusion can be drawn.

14.2 Points of Inflexion

The concavity of a curve can change, and the point where it changes often has a meaning. Suppose $f(t)$ is the total number of cases in an epidemic after t days. Then $f'(t)$ is the number of new cases per day. While $f'' > 0$ the daily count is growing. Once $f'' < 0$ the daily count is falling, even though the total still rises. The change from one to the other is the day on which the daily count is highest.

A **point of inflexion** is a point where the curve changes the direction of its bend: from concave up to concave down, or the reverse.

> DEF A **point of inflexion** is a point on a curve where the concavity changes. For a function that can be differentiated twice, $f''(x) = 0$ at every point of inflexion. But $f''(x) = 0$ alone does not guarantee one: the concavity must actually change.

CAUTION $f''(x) = 0$ does not by itself give an inflexion. For $f(x) = x^4$, $f''(0) = 0$, yet the curve is concave up on both sides: $x = 0$ is a minimum, not an inflexion. Always check that the concavity changes.

An inflexion can occur at any gradient, and the syllabus names two kinds. At a **stationary point of inflexion** the gradient is zero, as for $y = x^3$ at the origin. At a **non-stationary point of inflexion** the gradient is not zero, and the curve changes its bend while it is still rising or falling.

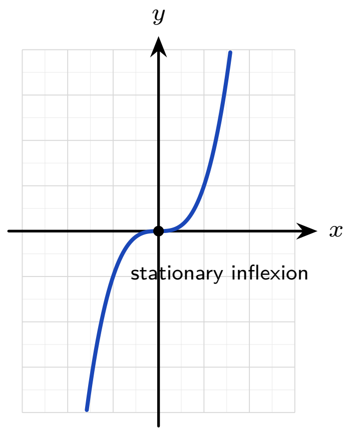


Figure 14.4 At the origin $y = x^3$ has both a horizontal tangent and a change of concavity, so this inflexion is also a stationary point.

To find the points of inflexion of a curve:

1. Find $f''(x)$ and solve $f''(x) = 0$.
2. Check that f'' changes sign at each solution, by testing a value on each side.
3. Substitute each confirmed value of x into f to find the point.
4. To name the kind, evaluate f' there: zero means stationary, non-zero means non-stationary.

Worked through. To find the point of inflexion of $f(x) = x^3 - 3x^2$, and whether it is stationary, differentiate twice:

$$f'(x) = 3x^2 - 6x, \qquad f''(x) = 6x - 6.$$

Set $f''(x) = 0$: $6x - 6 = 0 \implies x = 1$. For $x < 1$, $f'' < 0$ (concave down); for $x > 1$, $f'' > 0$ (concave up). The sign changes, so $x = 1$ gives a point of inflexion. Since $f(1) = -2$, the point is $(1, -2)$. The gradient there is $f'(1) = 3 - 6 = -3 \neq 0$, so this is a non-stationary inflexion: the curve changes its bend while it is falling.

14.2.1 Inflexions of other functions

Polynomials are not the only curves that bend both ways. Take $f(x) = \sin x$ and differentiate twice:

$$f'(x) = \cos x, \qquad f''(x) = -\sin x.$$

So $f''(x) = 0$ whenever $\sin x = 0$, that is at $x = 0, \pm\pi, \pm 2\pi, \dots$. At each of these the concavity changes, because $-\sin x$ changes sign every time the sine crosses the axis. The sine curve therefore has a point of inflexion at every multiple of π. None is stationary, since $f'(x) = \cos x = \pm 1$ there. Each inflexion is also the steepest point on its part of the curve. This follows from the definition: an inflexion of f is where f'' changes sign, so it is a turning point of f', where the gradient is greatest or least.

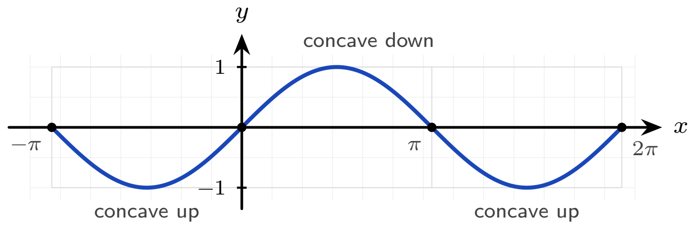


Figure 14.5 Every point where $y = \sin x$ crosses the axis is a point of inflexion: the bend reverses there, and the gradient is ± 1, the steepest the curve reaches. None of these inflexions is stationary.

Other familiar curves have no point of inflexion at all. For $y = e^x$, $f''(x) = e^x > 0$ for every x, so the curve is concave up everywhere. For $y = \ln x$, $f''(x) = -\dfrac{1}{x^2} < 0$ for every $x > 0$, so it is concave down across its whole domain. For $y = x^2$, $f''(x) = 2 > 0$ everywhere. In all three f'' never changes sign, so there is no inflexion to find.

EXAMPLE 14.2

Find the points of inflexion of $f(x) = e^{-x^2}$.

The chain rule gives $f'(x) = -2xe^{-x^2}$. Differentiating again needs the product rule:

$$f''(x) = -2e^{-x^2} + (-2x)(-2x)e^{-x^2} = (4x^2 - 2)\, e^{-x^2}.$$

Since $e^{-x^2} > 0$ for every x, the sign of f'' is decided by $4x^2 - 2$ alone. This is zero when $x^2 = \frac{1}{2}$, so $x = \pm\frac{1}{\sqrt{2}} \approx \pm 0.707$, and it changes sign at each: negative between the two values, positive outside them. Both are points of inflexion, and $f\left(\pm\frac{1}{\sqrt{2}}\right) = e^{-1/2} \approx 0.607$.

So the curve is concave down over the central hump and concave up in each tail. This is the bell shape of Ch. 12. The same calculation on $e^{-x^2/2}$, a constant multiple of the standard normal density, puts its inflexions at $x = \pm 1$: exactly one standard deviation from the mean.

14.2.2 Reading f, f' and f'' together

A common examination question shows one of these three graphs and asks about another. Below are $f(x) = x^3 - 3x$ and its two derivatives, drawn one above the other on the same x-axis. The vertical scales are not the same; only the zeros and the signs matter here.

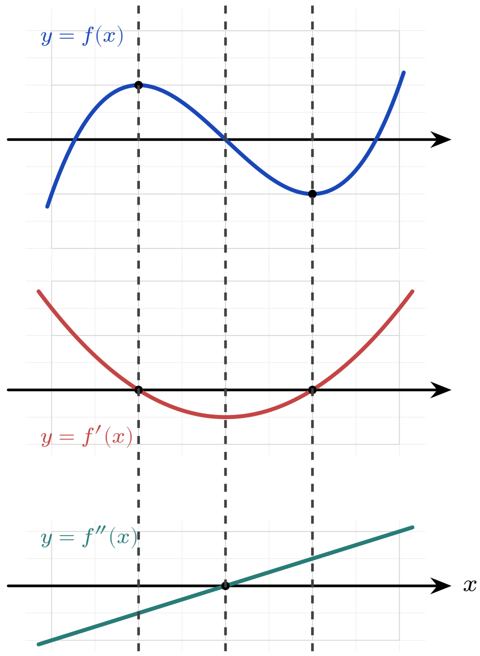


Figure 14.6 f, f' and f'' for $f(x) = x^3 - 3x$, stacked over a shared x-axis. Read down each dashed line: a turning point of f sits above a zero of f', and the inflexion of f sits above a turning point of f' and a zero of f''.

At every dashed line something happens in all three graphs at once. The three links are:

- A **turning point of** f sits above a **zero of** f'. At $x = -1$ the curve f has a maximum, and there f' crosses zero from positive to negative. At $x = 1$ the curve has a minimum, and there f' crosses zero from negative to positive.
- An **interval where** f **increases** sits above an **interval where** f' **is positive**, that is, where the graph of f' is above the axis.
- An **inflexion of** f sits above a **turning point of** f', and above a **zero of** f'' where f'' changes sign. In this example all three occur at $x = 0$.

Each link works in both directions, from f to f' and from f' back to f. So from any one of the three graphs you can sketch the other two.

YOUR TURN 14.2 Points of Inflexion

Finding points of inflexion

10. Find the point of inflexion of each curve, and state whether it is a stationary or a non-stationary point of inflexion.

(a) $y = x^3 - 6x^2 + 5$
(b) $y = x^3 + 6x^2 + 12x$
(c) $y = 2x^3 - 9x^2 + 12x$

11.

(a) Find the coordinates of the point of inflexion of $y = xe^x$.
(b) Find the x-coordinates of the points of inflexion of $y = 2x + \cos x$ for $0 \leq x \leq 2\pi$.
(c) Show that $y = \dfrac{x}{1 + x^2}$ has three points of inflexion, and find their coordinates.

When $f''(x) = 0$ is not enough

12.

(a) Show that $y = x^4 - 4x + 1$ has $\dfrac{d^2y}{dx^2} = 0$ at $x = 0$, and explain why there is no point of inflexion there.
(b) Determine, for $y = x^5$, whether the point where $f'' = 0$ is a point of inflexion, and if so whether it is stationary.
(c) Explain why $y = e^x + x^2$ has no point of inflexion.

Reading f, f' and f'' together

13. The graph of $y = f'(x)$ crosses the x-axis only at $x = -1$, from above to below, and at $x = 3$, from below to above. It has a single turning point, a minimum at $x = 1$. For the curve $y = f(x)$, state

(a) the x-coordinates of its stationary points, and the nature of each
(b) the interval on which f is decreasing
(c) the x-coordinate of its point of inflexion
(d) the interval on which it is concave up.

14. A function has $f'(x) = x^2(x - 3)$. Without finding $f(x)$,

(a) classify the stationary points of f
(b) find the x-coordinates of the points of inflexion of f, and state whether each is stationary or non-stationary.

Concept check

15. A student says: "*a point of inflexion is a point where the gradient is zero.*" Give a counter-example, and state the condition that defines a point of inflexion.

14.3 Optimisation

Optimisation means finding the largest or smallest value of a real quantity: the greatest volume, the least cost, the shortest time.

Without the derivative, you could only try inputs one at a time and compare the results. That can suggest an answer but cannot settle it, because between any two inputs you tried there are infinitely many you did not. The derivative replaces the search with an equation. At a maximum or a minimum inside the domain the curve is flat, so $f'(x) = 0$. Solving that one equation, together with checking any endpoints, gives the only candidates.

In practice, setting up the function is usually more work than the calculus.

Most optimisation questions follow the same five steps.

1. **Name the quantity to be optimised** and write an expression for it. At this stage it may contain more than one variable.
2. **Find the constraint**, the second equation linking those variables. It is given in the question, often as a fixed volume, a fixed length of material, or a fixed perimeter.
3. **Use the constraint to eliminate variables**, until the quantity is a function of one variable only. State the values that variable may take.
4. **Differentiate, set the derivative to zero, and solve.**
5. **Justify the nature of the point**, then answer the question that was actually asked.

Steps 3 and 5 are the ones to watch. Reducing to a single variable is where algebra often goes wrong, and skipping the justification can lose marks at the end.

TIP An optimisation question is not finished when you solve $f'(x) = 0$. Marks often depend on what follows: justify the nature of the point, using f'' or the sign of f', and then answer the question that was asked. If the question asks for the maximum volume, give the volume, not the value of x that produces it.

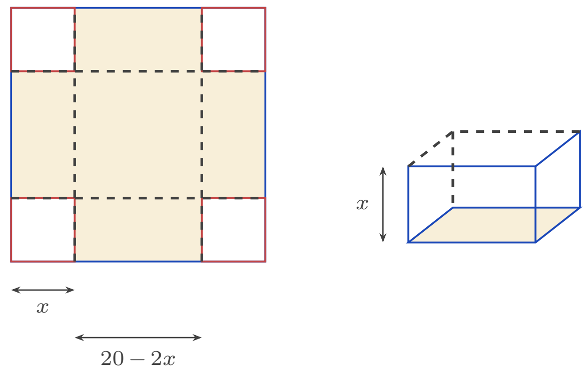


Figure 14.7 Cutting a square of side x from each corner of the card, then folding the flaps up. The base of the box measures $20 - 2x$ each way and its height is x, so $V = x(20 - 2x)^2$.

Worked through. To find the value of x that maximises the volume of the open box in the figure, made from a 20×20 cm square of card by cutting a square of side x from each corner and folding up the sides, note that after cutting the base is $(20 - 2x)$ by $(20 - 2x)$ and the height is x, so

$$V(x) = x(20 - 2x)^2, \qquad 0 < x < 10.$$

Differentiate with the product rule, then take out the common factor $(20 - 2x)$:

$$V'(x) = (20 - 2x)^2 + x \cdot 2(20 - 2x)(-2) = (20 - 2x)(20 - 6x).$$

Setting $V'(x) = 0$ gives $x = 10$ (rejected, no box) or $x = \frac{10}{3}$. Since $V' > 0$ before $\frac{10}{3}$ and $V' < 0$ after, this is a maximum by the first derivative test. The maximum volume is

$$V\left(\tfrac{10}{3}\right) = \frac{16000}{27} \approx 593 \text{ cm}^3.$$

The restriction $0 < x < 10$ comes from the geometry. If x is 0 the box has no height; if x is 10 it has no base.

EXAMPLE 14.3

A closed cylindrical can must hold $355\,\text{cm}^3$. Find the radius that minimises the surface area. Volume fixes the height: $\pi r^2 h = 355$, so $h = \dfrac{355}{\pi r^2}$. The surface area is two circles plus the curved side:

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{710}{r}.$$

Differentiate and set to zero:

$$\frac{dS}{dr} = 4\pi r - \frac{710}{r^2} = 0 \implies r^3 = \frac{710}{4\pi} \implies r \approx 3.84 \text{ cm}.$$

Since $\dfrac{d^2S}{dr^2} = 4\pi + \dfrac{1420}{r^3} > 0$, this is a minimum by the second derivative test. Either test earns the mark; choose the one whose working is shorter. Using the constraint to remove h is the step that reduces two variables to one.

CAUTION Setting $f' = 0$ finds only the best values *inside* an interval. If the variable is restricted, for example $0 < r \leq 3$, the best value may sit at an **endpoint** instead, where the derivative need not be zero. HL Always list the candidates: every stationary point in the interval, and both endpoints. Then compare their values and take the largest or the smallest.

EXAMPLE 14.4

For the can above, the manufacturer's machinery cannot roll a radius above 3 cm, so $0 < r \leq 3$. Find the radius that minimises the surface area now.

The unconstrained minimum at $r \approx 3.84$ is no longer allowed. Consider the derivative on $(0, 3]$:

$$\frac{dS}{dr} = 4\pi r - \frac{710}{r^2}.$$

Both terms increase as r increases, so $\dfrac{dS}{dr}$ is increasing on this interval. Testing the largest value is therefore enough: at $r = 3$, $12\pi - \frac{710}{9} \approx -41 < 0$. The derivative is negative at the right-hand end, and smaller still to the left, so it is negative throughout and S is decreasing across the whole interval. The minimum therefore sits at the endpoint $r = 3$, with $S = 18\pi + \frac{710}{3} \approx 293$ cm^2, even though $S'(3) \neq 0$.

Optimisation is not restricted to areas and volumes either. Any quantity that can be written as a differentiable function of one variable can be maximised or minimised the same way, and in applied work that function is often exponential.

EXAMPLE 14.5

After a dose is given, the concentration of a drug in the blood is modelled by $C(t) = 20t\, e^{-0.5t}$ mg L^{-1}, where $t \geq 0$ is the time in hours. Find when the concentration is greatest, and the maximum concentration.

The product rule gives

$$C'(t) = 20e^{-0.5t} + 20t(-0.5)e^{-0.5t} = 20e^{-0.5t}\,(1 - 0.5t)\,.$$

Since $e^{-0.5t}$ is never zero, $C'(t) = 0$ requires $1 - 0.5t = 0$, so $t = 2$ hours.

To justify that this is a maximum, look at the sign of C': the factor $e^{-0.5t}$ is always positive, so the sign is that of $1 - 0.5t$, which is positive for $t < 2$ and negative for $t > 2$. The concentration rises, then falls. The maximum is

$$C(2) = 40e^{-1} \approx 14.7 \text{ mg L}^{-1}.$$

The shape can be read from the two factors. For small t the factor t increases faster than $e^{-0.5t}$ decreases, so C rises. For large t the exponential factor decreases faster, so C falls. The maximum is where the two effects are equal, which is what $C'(t) = 0$ records.

YOUR TURN 14.3 Optimisation

Setting up and solving

16.

(a) Two numbers have a sum of 20. Find the maximum value of their product, and justify that it is a maximum.

(b) A rectangle has a perimeter of 40 cm. Find the dimensions giving the maximum area, and justify that it is a maximum.

(c) Find the minimum value of $S = x^2 + \dfrac{16}{x}$ for $x > 0$, justifying that it is a minimum.

17. A rectangular field is fenced on three sides, the fourth being a wall, using 100 m of fencing. Find the maximum area, and justify that it is a maximum.

Using a constraint to remove a variable

18. A closed box has a square base of side x cm and height h cm. Its total surface area is 150 cm^2.

(a) Show that the volume is $V = \dfrac{75x - x^3}{2}$, and state the possible values of x.

(b) Find the value of x that maximises the volume, and justify that it is a maximum.

(c) Find the maximum volume, and describe the shape of the box.

19. An open-topped cylindrical tank must hold 1000 cm^3. Show that its surface area is $S = \pi r^2 + \dfrac{2000}{r}$, and find the radius that minimises it.

20. Find the coordinates of the point on the curve $y = \sqrt{x}$ that is closest to the point $(4, 0)$, and the least distance. *(Hint: minimise the square of the distance)*

Exponential, logarithmic and trigonometric models

21.

(a) Find the minimum value of $y = x - \ln x$ for $x > 0$, justifying that it is a minimum.

(b) The concentration of a drug in the blood is $C(t) = 12t\, e^{-t/4}$ mg L^{-1}, where $t \geq 0$ is measured in hours. Find the time at which the concentration is greatest, and the maximum concentration.

22. GDC A rectangle has its base on the x-axis and its two upper vertices on the curve $y = \cos x$, symmetric about the y-axis, with $0 < x < \frac{\pi}{2}$. Show that its area is $A = 2x \cos x$, and find the value of x that maximises A.

Endpoints of a restricted domain

23. A rectangle has a perimeter of 40 cm, and one side, of length x cm, may be at most 8 cm long. Find the greatest possible area, and explain why it does not occur where $\dfrac{dA}{dx} = 0$.

24. For the drug in Question 21(b), blood samples are taken only during the first three hours, so $0 \le t \le 3$. Find the greatest concentration recorded in that time, and state when it occurs.

Concept check

25. A student is asked for the greatest value of $f(x) = x^3 - 3x$ for $0 \le x \le 3$, and writes: $f'(x) = 3x^2 - 3 = 0$, so $x = 1$ (as $x = -1$ is outside the interval), and the greatest value is $f(1) = -2$. Identify the two errors and find the greatest value.

14.4 Kinematics

Motion is change of position, so calculus describes it exactly. Let $s(t)$ be a particle's **displacement** from a fixed origin at time t. Its **velocity** is the rate of change of displacement, $v = \dfrac{ds}{dt}$, and its **acceleration** is the rate of change of velocity.

KEY · IB

$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

It helps to read the chain $s \to v \to a$ in ordinary words. Displacement answers *where*. Velocity answers *how fast the where is changing*. Acceleration answers *how fast the how-fast is changing*. In a car on a straight road, s is the signed distance from the starting point (the odometer reads distance travelled, which is different), the speedometer reads $|v|$, and the push you feel in your seat is a. Each quantity is the derivative of the one before it, which is why acceleration is a rate of a rate, and so a second derivative.

Velocity carries a sign, so it records direction as well as how fast the particle moves. On a straight line there are only two directions, and the sign says which one. **Speed** is the size of the velocity, written $|v|$, and it is never negative. A particle with $v = -8$ is moving in the negative direction at a speed of 8.

The particle is **momentarily at rest** (at rest for an instant) when $v = 0$. This is the stationary-point condition of §14.1 applied to displacement: a turning point on the graph of $s(t)$ is the instant the particle stops and reverses. The ball at the top of its flight is exactly this point.

Speed follows a separate rule, and it is better understood than memorised. Think of acceleration as a push. If the push acts in the direction the particle is already moving, the particle gains speed; if it acts against the motion, the particle loses speed. In symbols, the particle **speeds up when v and a have the same sign** and slows down when the signs differ. Notice that this does not depend on a being positive: a particle moving in the negative direction with a negative acceleration is speeding up, because the push and the motion agree.

CAUTION $a = 0$ does not mean the particle has stopped. Acceleration is the derivative of velocity, so $a = 0$ marks a stationary point of $v(t)$: an instant where the velocity may be greatest or least (a candidate, as with $f'(x) = 0$ in §14.1), not where it is zero. A car cruising at a steady $100\,\mathrm{km\,h^{-1}}$ has zero acceleration and is not stopped at all. The particle is at rest only when $v = 0$.

One quantity needs integration rather than differentiation: the **total distance travelled**, which keeps counting when the particle turns and comes back. Displacement over an interval is $\int v\,dt$ and distance travelled is $\int |v|\,dt$. Both belong to the same syllabus item as the derivatives above, and both are developed in Ch. 15.

Worked through. Take a particle whose displacement is $s(t) = t^3 - 12t^2 + 36t$ metres, for $t \geq 0$ seconds, and find when it is at rest and its acceleration at those times. Differentiate for velocity:

$$v(t) = 3t^2 - 24t + 36 = 3(t-2)(t-6).$$

The particle is at rest when $v = 0$: at $t = 2$ and $t = 6$ seconds. Acceleration is $a(t) = 6t - 24$. At $t = 2$, $a = -12\,\mathrm{m\,s^{-2}}$; at $t = 6$, $a = 12\,\mathrm{m\,s^{-2}}$. The sign change of velocity at each rest point means the particle reverses direction there.

Now ask whether the particle is speeding up at $t = 7$. There $v(7) = 3(5)(1) = 15 > 0$ and $a(7) = 6(7) - 24 = 18 > 0$. The two signs are the same, so the particle is speeding up: it moves in the positive direction and the acceleration acts the same way.

Not all motion is polynomial. Motion that repeats, such as a pendulum, a mass on a spring, a wave or a vibrating string, is often modelled by a sine or a cosine. When the acceleration is proportional to the displacement and acts in the opposite direction, so that $a = -k^2 s$ for a constant k, the motion is called **simple harmonic motion**. The particle is at rest at its greatest displacement, where the pull back towards the origin is strongest, and moves fastest as it passes through the origin, where the pull is zero.

EXAMPLE 14.6

A particle moves in a straight line with displacement $s = 4\sin 2t$ metres, for $t \geq 0$ seconds. Find its velocity and acceleration, the first time it is at rest, and show that $a = -4s$.

Differentiate twice, using the chain rule each time:

$$v = \frac{ds}{dt} = 8\cos 2t, \qquad a = \frac{dv}{dt} = -16\sin 2t.$$

The particle is at rest when $v = 0$, that is when $\cos 2t = 0$. The first solution is $2t = \frac{\pi}{2}$, so $t = \frac{\pi}{4} \approx 0.785$ s. At that instant $s = 4\sin\frac{\pi}{2} = 4$ m, the greatest displacement it reaches. Finally, since $s = 4\sin 2t$,

$$-4s = -4(4\sin 2t) = -16\sin 2t = a. \qquad \blacksquare$$

So $a = -k^2 s$ with $k = 2$, and the particle moves with simple harmonic motion: the further it is from the origin, the harder it is pulled back. The same relation describes a pendulum (for small swings), a mass on a spring and a sound wave.

TIP In kinematics questions, state the units with every answer: $\mathrm{m\,s^{-1}}$ for velocity, $\mathrm{m\,s^{-2}}$ for acceleration. A correct number without its unit may lose a mark.

TIP To find the greatest speed, compare $|v|$, not v. Because speed ignores direction, the greatest speed can occur where the velocity is most negative.

YOUR TURN 14.4 Kinematics

Velocity, acceleration and rest

26. A particle moves in a straight line. Its displacement is s metres and its velocity is v $\mathrm{m\,s^{-1}}$ at time t seconds, $t \geq 0$.

(a) Its displacement is $s = t^2 - 4t + 3$. Find when it is at rest, and its acceleration then.

(b) Its displacement is $s = 2t^3 - 9t^2 + 12t$. Find the times when it is at rest.

(c) Its displacement is $s = t^3 - 3t^2$. Find its velocity and acceleration at $t = 2$.

(d) Its velocity is $v = t^2 - 5t + 6$. Find when it is at rest, and its acceleration at those times.

27. A particle has velocity $v = 3t^2 - 12t + 9$ $\mathrm{m\,s^{-1}}$ at time t seconds, $t \geq 0$.

(a) Find the times when it is at rest.

(b) Find its acceleration at those times.

Speeding up or slowing down

28.

(a) For the particle in Question 27, determine whether it is speeding up or slowing down at $t = 1.5$, giving a reason.

(b) A second particle has velocity $v = t - 4$ m s^{-1} at time t seconds. State, with a reason, whether it is speeding up or slowing down at $t = 2$.

Speed

29. A particle has velocity $v = t^2 - 6t + 3$ m s^{-1} at time t seconds, for $0 \leq t \leq 5$.

(a) Find its speed at $t = 1$.

(b) Find the time at which its acceleration is zero. State, with a reason, whether the particle is at rest at that time.

(c) Find the greatest speed of the particle for $0 \leq t \leq 5$.

30. A particle has displacement $s = t^3 - 9t^2 + 24t$ metres at time t seconds, $t \geq 0$.

(a) Find the times when the particle is at rest.

(b) Find the time at which its acceleration is zero, and its speed at that time.

Other functions of time

31.

(a) A particle has displacement $s = 2\cos 3t$ metres, $t \geq 0$. Find v and a, and show that $a = -9s$.

(b) For the particle in part (a), find the first time after $t = 0$ at which it is at rest, and its displacement at that instant.

(c) A particle has velocity $v = 6e^{-0.5t}$ m s^{-1}. Find its acceleration at $t = 0$, and explain why the particle never comes to rest.

(d) A particle has displacement $s = t + 2\cos t$ metres, for $0 \leq t \leq 2\pi$. Find the times at which it is at rest.

Concept check

32. A student says: "*the acceleration is negative, so the particle is slowing down.*" State in one line when this is false.

33. Explain the difference between the *displacement* and the *distance travelled* over $0 \leq t \leq 5$, for a particle that reverses direction at $t = 3$.

14.5 Implicit Differentiation HL

So far every curve has been given as $y = f(x)$, with y written on its own. But an equation such as $x^2 + y^2 = 25$ also defines a curve, without y being isolated. **Implicit differentiation** finds the gradient of such a curve directly, without first solving for y. Every tangent and normal method of Ch. 13 then works on these curves too, including curves that are not the graph of a function.

The method is the chain rule. Treat y as an unknown function of x. A term in y then differentiates as usual, but it gains an extra factor $\dfrac{dy}{dx}$, because you are differentiating a function of y with respect to x. For example, $\dfrac{d}{dx}(y^2) = 2y\dfrac{dy}{dx}$.

The procedure has four steps.

1. Differentiate every term of the equation with respect to x, using the product rule where x and y are multiplied.
2. Collect the terms containing $\dfrac{dy}{dx}$ on one side.
3. Factorise $\dfrac{dy}{dx}$ out of those terms.
4. Divide to make $\dfrac{dy}{dx}$ the subject.

Worked through. For the circle $x^2 + y^2 = 25$, differentiate both sides with respect to x:

$$2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}.$$

At $(3, 4)$ the gradient is $-\frac{3}{4}$. The answer depends on both coordinates, and it must: the gradient changes as you move round the circle.

EXAMPLE 14.7

Find $\dfrac{dy}{dx}$ for $x^2 + xy + y^2 = 7$.

The middle term needs the product rule. Differentiating term by term:

$$2x + \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0.$$

Collect the $\dfrac{dy}{dx}$ terms:

$$\frac{dy}{dx}(x + 2y) = -(2x + y) \implies \frac{dy}{dx} = -\frac{2x + y}{x + 2y}.$$

Each of the four steps appears once: differentiate, collect, factorise, divide.

EXAMPLE 14.8

Find the equation of the tangent to the curve $x^2 + xy + y^2 = 7$ at the point $(1, 2)$.
First confirm the point lies on the curve: $1 + 2 + 4 = 7$. From the previous example,

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y} = -\frac{2 + 2}{1 + 4} = -\frac{4}{5} \quad \text{at } (1, 2).$$

The tangent is $y - 2 = -\frac{4}{5}(x - 1)$, or $4x + 5y = 14$. Implicit differentiation supplies the gradient. The rest is the usual equation of a line through a given point.

The same method handles exponential and trigonometric terms without any change. A term in y still gains a factor $\frac{dy}{dx}$, whatever function it sits inside.

EXAMPLE 14.9

The curve C has equation $e^y + xy = 1$. Show that $(0, 0)$ lies on C, and find the gradient there.
At $(0, 0)$: $e^0 + 0 = 1$, so the point lies on C.
Differentiate every term with respect to x. The first needs the chain rule and the second the product rule:

$$e^y \frac{dy}{dx} + \left(y + x\frac{dy}{dx}\right) = 0.$$

Collect the $\frac{dy}{dx}$ terms and factorise:

$$\frac{dy}{dx}(e^y + x) = -y \implies \frac{dy}{dx} = -\frac{y}{e^y + x}.$$

At $(0, 0)$ this gives $\frac{dy}{dx} = \frac{0}{1 + 0} = 0$, so the tangent there is horizontal.
Notice that this curve cannot be rearranged to give y in terms of the functions of this course: y appears both inside an exponential and multiplied by x. The curve still has a gradient at every point, and near $(0, 0)$ it still gives one y for each x. There is no expression in this course's functions to write down. Implicit differentiation is not a shortcut here; within the course, it is the most direct method available.

YOUR TURN 14.5 Implicit Differentiation HL

Finding the gradient function

34. Find $\frac{dy}{dx}$ for each curve.

(a) $x^2 + y^2 = 16$
(b) $xy = 12$
(c) $x^2 - y^2 = 9$
(d) $x^3 + y^3 = 6$

35. Each of these needs the product rule.

(a) Find $\frac{dy}{dx}$ for $x^2 + xy = 5$.
(b) Find $\frac{dy}{dx}$ for $x^2y + y^2 = 4$.

36. Find $\frac{dy}{dx}$ for each curve.

(a) $e^y = x^2 + y$
(b) $\sin y + x^2 = 4$
(c) $x \ln y = 4$

Gradients, tangents and normals

37.

(a) Find the gradient of $x^2 + y^2 = 169$ at the point $(5, -12)$.
(b) Find the equation of the tangent to $xy = 4$ at the point $(2, 2)$.
(c) On the curve $x^2 + 4y^2 = 16$, find the two points at which the tangent is horizontal.
(d) Explain why implicit differentiation gives a gradient in terms of both x and y, while differentiating $y = f(x)$ gives one in terms of x alone.

38.

(a) Find the gradient of $e^y + x = 3$ at the point $(2, 0)$.
(b) Show that $(1, 2)$ lies on the curve $x^3 + y^3 = 9$, and find the equation of the normal there.

39. Show that the point $(2, 1)$ lies on the curve $x^2 - 3xy + y^2 = -1$, and find the equations of the tangent and the normal at that point.

Concept check

40. A student differentiates $x^2 + y^2 = 25$ with respect to x and writes $2x + 2y = 0$. Identify the error, and find $\frac{dy}{dx}$.

14.6 Related Rates HL

When two quantities are linked by an equation and both change over time, their rates of change are linked too. **Related rates** problems use the chain rule to connect those rates. You differentiate the equation that relates the quantities with respect to time.

The method has three steps.

1. Write an equation relating the quantities.
2. Differentiate every term with respect to t, so that each variable contributes its own rate.
3. Substitute the values known at the instant in question, and solve for the rate you need.

If the equation contains a variable whose rate is neither given nor asked for, use a fixed relation from the shape, such as similar triangles, to remove that variable before differentiating.

TIP Substitute the given values only *after* differentiating. Putting $r = 5$ into the volume formula first turns a variable into a constant, and a constant has rate of change zero.

Worked through. Suppose air is pumped into a spherical balloon at $100\,\text{cm}^3\text{s}^{-1}$, and we need the rate at which the radius is increasing when $r = 5$ cm. Volume and radius are related by $V = \frac{4}{3}\pi r^3$. Differentiate with respect to t:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Substitute $\dfrac{dV}{dt} = 100$ and $r = 5$:

$$100 = 4\pi(25)\frac{dr}{dt} \implies \frac{dr}{dt} = \frac{1}{\pi} \approx 0.318 \text{ cm s}^{-1}.$$

The rate of change of volume is constant, but the radius increases more slowly as the balloon grows, because the same volume is spread over a larger surface.

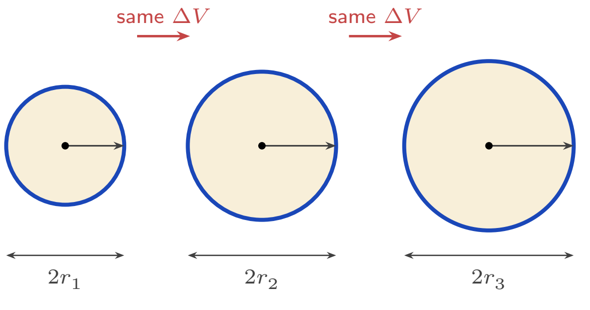


Figure 14.8 Equal volumes of air are added at each step, but the radius grows by less each time. The volume added is spread over the surface, and the surface area $4\pi r^2$ is itself growing, which is why $\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$ falls as r increases.

EXAMPLE 14.10

A 10 m ladder leans against a wall. The foot slides away from the wall at $0.6\,\mathrm{m\,s^{-1}}$. How fast is the top sliding down when the foot is 6 m from the wall?

Let x be the distance of the foot from the wall and y the height of the top. Then $x^2 + y^2 = 100$. Differentiate with respect to t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

When $x = 6$, $y = \sqrt{100 - 36} = 8$. With $\frac{dx}{dt} = 0.6$:

$$\frac{dy}{dt} = -\frac{6}{8}(0.6) = -0.45\ \mathrm{m\,s^{-1}}.$$

The negative sign means the top moves downward, as expected.

EXAMPLE 14.11

A water tank is an inverted cone with its vertex at the bottom, as tall as it is wide: at every level, the water surface has radius $r = \frac{h}{2}$, where h is the water depth. Water flows in at $2\,\mathrm{m^3\,min^{-1}}$. How fast is the depth rising when $h = 3$ m?

Write the volume in terms of h alone, using the shape constraint $r = \frac{h}{2}$:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}.$$

Differentiate with respect to time:

$$\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt} \implies 2 = \frac{9\pi}{4}\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{8}{9\pi} \approx 0.283\ \mathrm{m\,min^{-1}}.$$

Remove r *before* differentiating. Differentiating an expression that still contains two linked variables is a common source of error.

The algebra also shows the physics: the same inflow raises the level more slowly as the cone widens.

The quantities linked need not be lengths and volumes. When the link is an angle, the relating equation is trigonometric, and the method is unchanged.

EXAMPLE 14.12

A balloon rises vertically at $3\,\mathrm{m\,s^{-1}}$ from a point 40 m from an observer. Find the rate at which the angle of elevation of the balloon is increasing when the balloon is 40 m high.

Let h be the height of the balloon and θ the angle of elevation. The observer's distance is fixed at 40 m, so

$$\tan\theta = \frac{h}{40}.$$

Differentiate both sides with respect to t. The left side needs the chain rule:

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{40}\,\frac{dh}{dt}.$$

When $h = 40$ the triangle is isosceles, so $\theta = \frac{\pi}{4}$ and $\sec^2\theta = 2$. Substituting this and $\dfrac{dh}{dt} = 3$:

$$2\,\frac{d\theta}{dt} = \frac{3}{40} \implies \frac{d\theta}{dt} = \frac{3}{80} = 0.0375\ \text{rad s}^{-1}.$$

The balloon rises at a constant rate, but the angle does not increase at a constant rate. As the balloon climbs, $\sec^2\theta$ grows, so $\dfrac{d\theta}{dt}$ falls: the observer's head turns more and more slowly, even though the balloon never slows down.

YOUR TURN 14.6 Related Rates HL

One formula, one rate

41.

(a) A square's side grows at $2\,\text{cm s}^{-1}$. Find the rate of change of its area when the side is 5 cm.

(b) A circle's radius grows at $3\,\text{cm s}^{-1}$. Find the rate of change of its area when $r = 4$ cm.

(c) A cube's side grows at $1\,\text{cm s}^{-1}$. Find the rate of change of its volume when the side is 10 cm.

(d) A circle's radius grows at 0.4 cm s^{-1}. Find the rate of increase of its circumference.

42. A spherical snowball melts so that its volume decreases at $20\,\text{cm}^3\,\text{min}^{-1}$. Find the rate of change of its radius when $r = 8$ cm, and explain what the sign of your answer means.

Pythagoras and similar triangles

43. A ladder 13 m long leans against a vertical wall. Its top slides down the wall at $0.5\,\mathrm{m\,s^{-1}}$. Find the rate at which the foot moves away from the wall when the top is 5 m above the ground.

44. Water is poured at $20\,\mathrm{cm^3 s^{-1}}$ into a cone whose vertex points downwards. The water depth h is always twice the radius r of the water surface.

(a) Show that the volume of water is $V = \frac{2}{3}\pi r^3$.

(b) Find the rate at which r increases when $r = 5$ cm.

45. A conical tank, vertex down, is 6 m deep and has a top radius of 2 m. Water drains out at $0.5\,\mathrm{m^3\,min^{-1}}$. Find the rate of change of the water depth when the depth is 3 m, and interpret its sign. *(Hint: use similar triangles to write r in terms of h)*

Angles

46. An aircraft flies horizontally at a height of 3 km, at $0.2\,\mathrm{km\,s^{-1}}$, directly towards an observer on the ground. Find the rate of change of the angle of elevation of the aircraft when its horizontal distance from the observer is 4 km.

47. A searchlight rotates at $0.2\,\mathrm{rad\,s^{-1}}$ and its beam sweeps along a straight wall 50 m away. Find the speed of the spot of light along the wall when the beam makes an angle of $\frac{\pi}{6}$ with the perpendicular to the wall.

Concept check

48. The radius of a circle grows at $0.5\ \mathrm{cm\,s^{-1}}$. To find the rate of change of its area when $r = 6$ cm, a student writes $A = \pi(6)^2 = 36\pi$, so $\dfrac{dA}{dt} = 0$. Identify the error and find the correct rate.

Chapter 14 · Derivative Applications

Reflections

TOK This chapter is full of conditions that are necessary but not sufficient (they must hold, but they are not enough on their own): $f'(x) = 0$ holds at every smooth maximum, and, for a twice-differentiable function, $f''(x) = 0$ at every point of inflexion, yet neither condition guarantees one. What kind of knowledge is a necessary condition? Does knowing where something must be count as knowing where it is?

TOK Calculus makes exact claims about idealised objects. A ball is motionless at the top of its flight, but only for an instant, and an instant has no duration. The can that uses least metal has its height equal to its diameter, but real cans are taller and narrower, because the model ignores printing, shipping and the size of a hand. When a model is exact and its conclusion still differs from the world, which one is wrong: the model, or our reading of it?

IM The idea that nature minimises something appears across physics under many names. In seventeenth-century France, Pierre de Fermat proposed that light travels the path taking the least time, which explains refraction. Euler and Lagrange later generalised this into the *calculus of variations*, the study of which curve or path makes a quantity smallest, and the *principle of least action*, which restates much of classical mechanics as finding where a quantity is stationary. The belief that the world is in some sense optimal has produced a great deal of physics.

RW Many modern machine-learning models are trained by *gradient descent*, which is this chapter's idea at scale. A model has millions of adjustable numbers, and how wrong its predictions are is a function of those numbers; training means making that function as small as possible. With millions of variables the minimum cannot be found directly, so the computer measures the gradient at its current point, takes a small step in the direction that makes the function smaller, and repeats. As the gradient approaches zero the steps shrink and the values approach a minimum. That is $f' = 0$, applied billions of times.

End-of-Chapter Problems
Chapter 14: Derivative Applications

1. A stone dropped in a pond sends out a circular ripple whose radius grows at $0.5\,\mathrm{m\,s^{-1}}$. Find the rate at which the enclosed area is increasing when the radius is 8 m. [3]

2. For $f(x) = x^4 - 4x^3 + 6x^2$, show that $f''(1) = 0$ but that the point $(1, 3)$ is *not* a point of inflexion. [4]

3. The curve $\sin y + x^2 = 1$ passes through the point $(1, 0)$. Find the gradient of the curve at that point. [4]

4. A particle's velocity is $v(t) = 2t^2 - 14t + 20\,\mathrm{m\,s^{-1}}$, $t \geq 0$.

(a) Find the times when the particle is at rest. [2]

(b) Find the minimum velocity and the time at which it occurs. [3]

5. A company's profit from selling x units is $P(x) = -2x^2 + 120x - 1000$ dollars. Find the value of x that maximises the profit, justifying that it is a maximum, and state the maximum profit. [5]

6. A curve has equation $x^2 + xy + y^2 = 3$.

(a) Find $\dfrac{dy}{dx}$ in terms of x and y. [3]

(b) Find the equation of the tangent at the point $(1, 1)$. [3]

7. Consider $g(x) = x + \dfrac{4}{x}$ on the interval $1 \leq x \leq 3$.

(a) Find the stationary point of g in the interval and classify it. [4]

(b) Find the maximum value of g on the interval, and state where it occurs. [2]

8. Consider $f(x) = x + 2\cos x$ on the closed interval $0 \leq x \leq 2\pi$.

(a) Find the stationary points in the interval and classify them. [4]

(b) Find the global maximum and minimum of f on the interval. [3]

9. Consider $f(x) = e^x - 2x$.

(a) Find $f'(x)$ and the coordinates of the stationary point. [3]

(b) Determine the nature of the stationary point. [2]

(c) Explain why f has no point of inflexion. [2]

10. Consider $f(x) = x^3 - 3x^2 - 9x + 5$.

(a) Find and classify the stationary points. [4]

(b) Find the point of inflexion. [2]

(c) State the interval on which f is decreasing. [2]

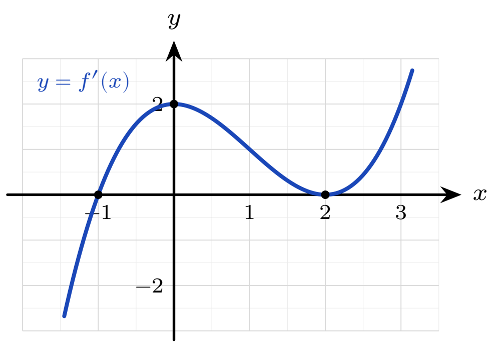
11. A particle moves with displacement $s(t) = t^3 - 9t^2 + 15t$ metres, $t \geq 0$.

(a) Find the velocity and the times when the particle is at rest. [3]

(b) Find the acceleration when $t = 1$. [2]

(c) Determine whether the particle is speeding up or slowing down at $t = 1.5$. [3]

12. The graph of $y = f'(x)$, the derivative of a function f, is shown for $-1.5 \leq x \leq 3$. It meets the x-axis only at $x = -1$ and at $x = 2$, and its turning points are at $x = 0$ and $x = 2$.



(a) Find the x-coordinate of the local minimum point of the graph of f. Justify your answer. [2]

(b) Explain why the graph of f has a stationary point at $x = 2$ that is not a turning point. [2]

(c) Write down the x-coordinates of the points of inflexion of the graph of f, and the interval on which the graph of f is concave down. [2]

13. The curve $y = x^3 + ax^2 + bx$ has stationary points at $x = 1$ and $x = -2$.

(a) Find the values of a and b. [4]

(b) Classify each stationary point. [3]

14. For the curve $x^2 + y^2 = 25$, use implicit differentiation twice to show that $\dfrac{d^2y}{dx^2} = -\dfrac{25}{y^3}$. [6]

15. A street lamp is 5 m tall. A person of height 1.8 m walks away from it at $1.2\,\mathrm{m\,s^{-1}}$. Find the rate at which the length of the person's shadow is increasing. [6]

16. A water tank is an inverted cone in which the water surface radius is always $\frac{3}{4}$ of the depth h. Water is pumped in at $1.5\,\mathrm{m^3\,min^{-1}}$. Find the rate at which the depth is rising when $h = 2$ m. [6]

17. A 5 m ladder slides down a wall, its foot moving away at a constant $0.5\,\text{m s}^{-1}$.

(a) Find the rate at which the top descends when the foot is 3 m from the wall. [3]

(b) Find the rate at which the area of the triangle formed by the ladder, wall, and ground is changing at that instant. [4]

18. GDC Let D be the distance from the origin to a point (x, e^x) on the curve $y = e^x$.

(a) Write down an expression for D^2 in terms of x. [1]

(b) Show that D^2 is stationary where $x + e^{2x} = 0$. [2]

(c) Find the coordinates of the point on the curve closest to the origin, and the least distance, giving your answers to 3 significant figures. [3]

19. GDC A closed cylindrical can must hold $500\,\text{cm}^3$.

(a) Show that the surface area is $S = 2\pi r^2 + \dfrac{1000}{r}$. [3]

(b) Find the radius that minimises S, and justify it is a minimum. [4]

20. An open rectangular tank with a square base of side x must hold $32\,\text{m}^3$.

(a) Show that the surface area (base + four sides) is $A = x^2 + \dfrac{128}{x}$. [3]

(b) Find the value of x that minimises the material used. [4]

21. GDC The population of a colony, in thousands, is modelled by $P(t) = 40t\,e^{-t/5}$ for $t \geq 0$, where t is measured in years.

(a) Show that $P'(t) = 8e^{-t/5}(5 - t)$. [3]

(b) Find the time at which the population is largest, and the largest population, giving the population to three significant figures. [4]

22. A particle moves with displacement $s(t) = t^3 - 12t^2 + 36t - 10$ metres, for $0 \leq t \leq 8$ seconds.

(a) Find the times at which the particle is at rest. [3]

(b) Find its acceleration at each of those times. [2]

(c) Find the intervals of time during which the particle's *speed* is increasing. [3]

23. Consider $f(x) = e^{-x^2/2}$.

(a) Find $f'(x)$, and show that $f''(x) = (x^2 - 1)\,e^{-x^2/2}$. [5]

(b) Find the exact coordinates of the points of inflexion, and state the interval on which the curve is concave down. [3]

24. Consider $f(x) = x^2 e^{-x}$.

(a) Find $f'(x)$ and the coordinates of the two stationary points. [4]
(b) Use the second derivative to determine the nature of each stationary point. [3]
(c) Find the exact x-coordinates of the points of inflexion. [2]

25. A particle moves along the curve $y = x^2$ so that $\dfrac{dx}{dt} = 2$ at all times. Find the rate at which its distance from the origin is increasing as it passes through the point $(1, 1)$. [7]

26. A cone is inscribed in a sphere of radius R. Show that the cone of maximum volume has height $h = \frac{4R}{3}$. [7]

27. A particle moves in a straight line with displacement $s(t) = 3\sin 2t$ metres, $t \geq 0$ seconds.

(a) Find expressions for the velocity and the acceleration. [2]
(b) Show that $a = -4s$. [2]
(c) Find the first two times at which the particle is at rest. [3]
(d) State the maximum speed of the particle, and the displacement at which it occurs. [3]

28. GDC A particle moves in a straight line with displacement $s = 5\sin t + 0.5t^2$ metres at time t seconds, for $0 \leq t \leq 6$.

(a) Find expressions for the velocity v and the acceleration a. [2]
(b) Find the times at which the particle is at rest, giving your answers to 3 significant figures. [3]
(c) Find the greatest speed of the particle for $0 \leq t \leq 6$, giving your answer to 3 significant figures. [3]
(d) Determine whether the particle is speeding up or slowing down at $t = 3$, giving a reason. [2]

29. A gutter is made by bending a long strip of metal of width 30 cm into three equal sections of 10 cm. The two outer sections are turned up at an angle θ to the horizontal, where $0 < \theta < \frac{\pi}{2}$. The cross-sectional area is $A = 100\sin\theta\,(1 + \cos\theta)$.

(a) Find $\dfrac{dA}{d\theta}$. [3]
(b) Show that A is stationary when $2\cos^2\theta + \cos\theta - 1 = 0$, and find the value of θ that maximises A. [4]
(c) Find the maximum cross-sectional area. [2]

30. Consider $f(x) = \dfrac{\ln x}{x^2}$ for $x > 0$.

(a) Show that $f'(x) = \dfrac{1 - 2\ln x}{x^3}$. [3]

(b) Find the exact coordinates of the stationary point and determine its nature. [4]

(c) Show that $f''(x) = \dfrac{6\ln x - 5}{x^4}$, and hence find the exact x-coordinate of the point of inflexion. [4]

31. The curve C has equation $x^3 + y^3 = 9xy$.

(a) Verify that the point $(2, 4)$ lies on C. [2]

(b) Use implicit differentiation to show that $\dfrac{dy}{dx} = \dfrac{x^2 - 3y}{3x - y^2}$. [4]

(c) Find the gradient of C at $(2, 4)$. [2]

(d) Find the equation of the tangent to C at $(2, 4)$. [3]

32. A lighthouse stands 2 km from a straight shoreline, and its beam rotates at one revolution per minute. Let θ be the angle between the beam and the perpendicular from the lighthouse to the shore, and let x be the distance from the foot of that perpendicular to the spot of light.

(a) Write x in terms of θ, and state the value of $\dfrac{d\theta}{dt}$ in radians per minute. [2]

(b) Show that $\dfrac{dx}{dt} = 4\pi \sec^2\theta$. [3]

(c) Find the speed of the spot of light along the shore when $\theta = 30^\circ$. [3]

(d) Describe what happens to this speed as $\theta \to 90^\circ$, and explain why. [2]

33. An open-topped box is made from a 16×10 cm rectangle of card by cutting a square of side x cm from each corner and folding up the sides.

(a) Show that the volume is $V = x(16 - 2x)(10 - 2x)$, and state the possible values of x. [3]

(b) Find $\dfrac{dV}{dx}$ and show that it is zero at $x = 2$ and $x = \frac{20}{3}$. [3]

(c) Explain why $x = \frac{20}{3}$ must be rejected. [2]

(d) Find the maximum volume, justifying that it is a maximum. [4]

34. GDC A particle moves in a straight line with displacement $s(t) = 12\left(1 - e^{-t/3}\right)$ metres, for $t \geq 0$ seconds.

(a) Find expressions for the velocity and the acceleration. [3]

(b) Write down the initial velocity, and show that the velocity is never zero. [2]

(c) Find the displacement and the velocity at $t = 3$, each to three significant figures. [2]

(d) State what happens to s and to v as $t \to \infty$, and interpret the limiting value of s. [3]

(e) Find the exact time at which the speed has fallen to half its initial value. [3]

35. Consider $f(x) = x^2 \ln x$ for $x > 0$.

(a) Find $f'(x)$, and factorise it. [2]

(b) Find the exact coordinates of the stationary point. [3]

(c) Find $f''(x)$, and use it to classify that point. [3]

(d) Find the exact coordinates of the point of inflexion, and show that the concavity changes there. [4]

(e) Describe the behaviour of $f(x)$ as $x \to 0^+$ and as $x \to \infty$. [3]

Challenge Questions

36. GDC **Modelling the best can.** The notes show that a closed cylinder of fixed volume uses least metal when its height equals its diameter. Real drink cans are taller than that. This investigation rebuilds the model around cost instead of area, and asks how far the improved model explains the cans on a shop shelf. Throughout, a closed can of radius r and height h holds a fixed volume V.

(a) Suppose the metal for the two ends costs k times as much per unit area as the metal for the curved side, where $k > 0$. Show that the cost of the can is proportional to $C = 2k\pi r^2 + \dfrac{2V}{r}$. Show that C has a minimum when $h = 2kr$, and hence that the cheapest can has $\dfrac{h}{d} = k$, where d is the diameter. [5]

(b) Now let the metal cost the same everywhere, but suppose each end is cut from its own square of side $2r$, and the corners are scrap that must still be paid for. Show that the cheapest can has $\dfrac{h}{d} = \dfrac{4}{\pi}$. [4]

(c) If instead the ends are cut from one sheet with their centres on a triangular grid, each disc uses a regular hexagon whose inscribed circle is the disc. Show that such a hexagon has area $2\sqrt{3}\,r^2$, and that the cheapest can then has $\dfrac{h}{d} = \dfrac{2\sqrt{3}}{\pi}$. [3]

(d) A typical 330 ml drink can is about 11.5 cm tall and 6.6 cm in diameter. Find the value of k in the model of part (a) that would make this shape the cheapest, and compare it with the values implied by parts (b) and (c). [2]

(e) Take $k = 1$ and write $h = xd$ with $x > 0$. Show that the surface area is

$$S = (2\pi)^{1/3}\,V^{2/3}\,\frac{1+2x}{x^{2/3}}.$$

Find the percentage of extra metal used by the drink-can shape of part (d) compared with the shape $x = 1$. Use the derivative $\dfrac{dS}{dx}$ at $x = 1$ to explain why a shape some way from the optimum costs only a little more. [4]

(f) List other features of a real can that a better model would include, and describe how you would investigate one of them. [3]

37. **Closest approach.** Finding the point of a curve nearest to a fixed point is an optimisation problem, and its answer has a geometric description. This investigation finds that description, states exactly when it holds, and uses it to count the normals that can be drawn to a parabola from a given point.

(a) Let $(x, \sqrt{x})$ be a point on $y = \sqrt{x}$ and let D be its distance from $(4, 0)$. Show that $D^2 = x^2 - 7x + 16$. [3]

(b) Explain why minimising D^2 gives the same answer as minimising D, and why using D^2 is easier. [2]

(c) Find the point on the curve closest to $(4, 0)$, and the least distance. [4]

(d) Show that at this closest point the line joining it to $(4, 0)$ is perpendicular to the tangent to the curve. [3]

(e) Let f be differentiable on an open interval, and let (a, b) be a point not on the curve $y = f(x)$. Suppose the distance from (a, b) to $(x, f(x))$ has a local minimum at a point x_0 of the interval. Show that

$$(x_0 - a) + (f(x_0) - b)f'(x_0) = 0,$$

and deduce that the line from (a, b) to $(x_0, f(x_0))$ is perpendicular to the tangent there. (This line is the *normal* to the curve at that point.) [4]

(f) Show that the conclusion of part (e) can fail when its conditions are dropped, using: (i) the curve $y = |x|$ and the point $(0, -1)$; (ii) the ray $y = x$, $x \geq 0$, and the point $(-2, 0)$. In each case find the closest point and state which condition fails. [3]

(g) Use the condition in part (e) to show that the number of normals to $y = x^2$ that pass through the point $(0, b)$ is three if $b > \frac{1}{2}$ and one if $b \leq \frac{1}{2}$. [3]

(h) The value $b = \frac{1}{2}$ in part (g) is the radius of curvature of $y = x^2$ at its vertex (Question 38(e)). Investigate this connection, and the curve that separates the points from which three normals to $y = x^2$ can be drawn from those from which only one can. [2]

38. Quantifying curvature. A straight line does not bend; a small circle bends sharply. This investigation builds a number that measures how sharply a curve bends at a point. It is defined by

$$\kappa = \frac{|y''|}{\left(1 + (y')^2\right)^{3/2}},$$

and κ is called the *curvature*.

(a) Show that every straight line has $\kappa = 0$ at every point, and explain why the definition must give this. [3]

(b) For $y = x^2$, find κ as a function of x. Show that the curvature is greatest at the vertex, and find its value there. [4]

(c) Show that $\kappa \to 0$ as $x \to \pm\infty$ for $y = x^2$, and explain what this says about the shape of a parabola far from its vertex. [3]

(d) Use implicit differentiation on $x^2 + y^2 = R^2$ to show that a circle of radius R has $\kappa = \dfrac{1}{R}$ at every point. Explain why a constant curvature is exactly what you should expect. [5]

(e) The *radius of curvature* is defined as $\rho = \dfrac{1}{\kappa}$. Find ρ for $y = x^2$ at the origin, and state the circle that best matches the parabola there. [3]

(f) The second derivative alone will not do as a measure of bending. For $y = x^2$, $y'' = 2$ at every point, yet the curve clearly bends more sharply at the vertex than far out along an arm. State what the denominator $\left(1 + (y')^2\right)^{3/2}$ corrects for, and use your answer to part (b) to support it. [3]

(g) Show that $\kappa = 0$ at every point of inflexion of a twice-differentiable curve. Then explain why a curve with $\kappa = 0$ at one point need not be a straight line, and give an example. [3]

(h) The curve $y = f(x)$ is stretched vertically by a factor $k > 0$, giving $y = k\,f(x)$. At a point where $f'(x) = 0$, show that the curvature of the stretched curve is k times the curvature of the original. Show also that a translation changes κ nowhere. What does this say about judging how sharply a curve bends from a graph drawn with unequal scales on the two axes? [4]

(i) Engineers join a straight railway track to a circular arc of radius R by a *transition curve*, along which the curvature increases steadily from 0 to $\frac{1}{R}$. For the cubic $y = cx^3$ with $c > 0$, show that $\kappa \approx 6cx$ when x is small and positive, and find the value of c for which this approximation reaches $\frac{1}{R}$ at $x = L$. Then investigate the curve whose curvature is exactly proportional to the distance travelled along it (the *clothoid*, or Euler spiral), and how closely the cubic approximates it. [3]

39. Filling a vessel. Water is poured at a constant rate Q into an empty vessel whose horizontal cross-section at height h above the lowest point has area $A(h)$. The shape of the vessel decides how the water level rises. This investigation finds how, and then reverses the question: given a required pattern of rising, what shape must the vessel have?

(a) The water between heights h and $h + \delta h$ forms a thin slice. Explain why its volume is approximately $A(h)\,\delta h$, and hence why $\dfrac{dV}{dh} = A(h)$. Deduce that $\dfrac{dh}{dt} = \dfrac{Q}{A(h)}$. [3]

(b) For a cylinder of radius R, show that the level rises at a constant rate. For a sphere of radius R filled from its lowest point, show that $A(h) = \pi(2Rh - h^2)$ for $0 \le h \le 2R$, and find the height at which the level rises most slowly. [4]

(c) Show that $\dfrac{d^2h}{dt^2} = -\dfrac{Q^2 A'(h)}{A(h)^3}$. [3]

(d) Hence state and prove a rule linking the concavity of the graph of h against t to whether the vessel is widening or narrowing at the water surface. Apply it to the sphere in part (b). [3]

(e) A bowl is formed by rotating the curve $y = cx^2$ $(c > 0)$ about the y-axis. Show that $A(h) = \dfrac{\pi h}{c}$, and verify that $h = \sqrt{\dfrac{2Qct}{\pi}}$ satisfies $\dfrac{dh}{dt} = \dfrac{Q}{A(h)}$ with $h = 0$ at $t = 0$. [4]

(f) An ancient water clock empties through a small hole in its base. When the water depth is h, the water leaves at a rate proportional to $\sqrt{h}$, so that $\dfrac{dh}{dt} = -\dfrac{\lambda\sqrt{h}}{A(h)}$ for a constant $\lambda > 0$. Find how $A(h)$ must depend on h for the level to fall at a constant rate, so that equal marks on the side show equal times, and investigate the shape of such a vessel. [3]