CALCULUS

CHAPTER 13

Limits & Derivatives

13

SYLLABUS 5.1–5.4 5.6 5.7 5.12 5.13 5.15

A car's speedometer reads 80 km/h. If you ask what that number means, you meet a problem that took mathematicians many centuries to settle.

Speed is distance divided by time. But the speedometer shows your speed *right now*, at a single instant. In one instant no time passes, and no distance is covered. Speed at an instant therefore appears to be $0 \div 0$, which has no value. How can the needle point to 80?

The solution is to stop asking about the instant, and to ask instead about short intervals around it.

Measure your average speed over the next hour. Then over the next minute. Then over the next second, and then over the next hundredth of a second. Each interval is shorter than the one before, and each average lies closer to one particular number. That number is the speed at the instant. The averages approach it, even though no calculation made at the instant itself can reach it.

This idea is called the **limit**. When a value cannot be reached directly, you find it by asking what nearby values approach. The rest of this chapter, and most of calculus, is built on it.

The **derivative** follows from the limit. It measures the exact rate at which a quantity changes: how steep a curve is at one point, or how fast an apple is falling at the moment it lands. This chapter builds the derivative from the limit, then develops methods for calculating it quickly.

By the end of this chapter you will evaluate limits, decide when a function is continuous and differentiable, and differentiate from first principles. You will apply the product, quotient and chain rules, differentiate the standard functions of the course, and find the equations of a tangent and a normal. You will evaluate limits of the form $\frac{0}{0}$ and $\frac{\infty}{\infty}$ with l'Hôpital's rule. You will also differentiate more than once, and meet the second derivative and the notation for every higher derivative. Calculus begins here because a derivative answers two questions at once. Geometrically it is the gradient of a curve at a point. Physically it is a rate of change. Most later applications use one of those two readings.

13.1 Limits

A **limit** is the value a function approaches as its input approaches some point. It does not matter what the function does at that point itself. We write

$$\lim_{x\to a} f(x) = L$$

to mean: as x gets **arbitrarily close** (as close as you like) to a, $f(x)$ gets arbitrarily close to L. The value of f at a itself plays no part, so a limit can exist at a point where the function is undefined.

For most functions at most points, the limit is found by direct substitution. If f is defined at a and its graph passes through that point unbroken, then the value it approaches is simply the value it takes:

$$\lim_{x\to 2}(x^2 + 3x) = 2^2 + 3(2) = 10.$$

The cases that need a method are the ones where substitution fails. When substitution gives $\frac{0}{0}$, factorise the numerator and the denominator and cancel the common factor. The cancelled function agrees with the original everywhere except at the hole, so its value there is the limit.

Worked through. To find $\lim_{x\to 3} \frac{x^2-9}{x-3}$, first try substituting $x = 3$. It gives $\frac{0}{0}$, which is undefined: the function has a hole at $x = 3$. But for every $x \neq 3$ we can factor and cancel:

$$\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x + 3.$$

Away from the hole the function is simply $x + 3$, so as $x \to 3$ it approaches 6:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = 6.$$

At $x = 3$ the function has no value at all, yet 6 is the value it approaches. The difference between "approaches" and "equals" is what a limit measures.

The form $\frac{0}{0}$ is called **indeterminate**. It does not mean the limit fails to exist. It means more work is needed.

One limit of this kind must be memorised, because the derivatives of the trigonometric functions depend on it:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

When there is no algebraic method, you can *estimate* a limit. Calculate the function at values closer and closer to the target, from both sides. Here is the result for this one, with x in radians:

x	-0.1	-0.01	0.01	0.1
$\dfrac{\sin x}{x}$	0.99833	0.99998	0.99998	0.99833

Both sides approach 1. A table or a zoomed-in graph is a valid way to **conjecture** (guess, before proving) the value of a limit. On a calculator paper it is often the fastest way to check an algebraic answer.

The phrase "from both sides" needs its own notation. Write $x \to a^-$ when x approaches a through values *less* than a (from the left), and $x \to a^+$ when it approaches through values *greater* than a (from the right). The value that $f(x)$ approaches from each side is a **one-sided limit**:

$$\lim_{x\to a^-} f(x) \quad \text{(left-hand limit)}, \qquad \lim_{x\to a^+} f(x) \quad \text{(right-hand limit)}.$$

DEF The limit $\lim\limits_{x\to a} f(x)$ exists only when the left-hand and right-hand limits both exist and are **equal**. Their common value is the limit.

The two graphs below show both cases. On the left, the graph has a jump at $x = 2$. Moving along the curve from the left, the height approaches 1; moving from the right, it approaches 3. The one-sided limits differ, so $\lim_{x\to 2} f(x)$ does not exist. On the right, the graph has only a hole at $x = 2$. Both sides approach the same height, 3, so the limit exists and equals 3, even though the function has no value at $x = 2$.

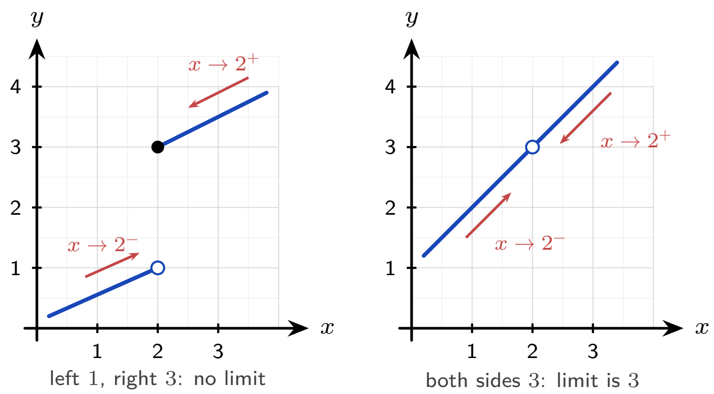

Figure 13.1 Left and right-hand limits. An open circle marks a point that is not on the graph. **Left:** at a jump the two sides approach different heights, so the limit does not exist. **Right:** at a hole both sides approach the same height, so the limit exists although $f(2)$ is undefined.

13.1.1 Reading a limit off the graph

A graph often shows why a limit takes the value it does. The two fractions $\dfrac{\sin x}{x}$ and $\dfrac{\cos x - 1}{x}$ both give $\frac{0}{0}$ at $x = 0$, yet they approach different values. The reason is how fast each numerator approaches 0 compared with the denominator x.

Sketch $y = \sin x$ and $y = x$ on the same axes. Near the origin the two graphs are almost the same: at $x = 0.1$, $\sin x = 0.0998$. The numerator and the denominator are almost equal, so their ratio approaches 1.

Now sketch $y = \cos x - 1$. Near the origin this graph is almost flat, because $\cos x$ has a maximum at $x = 0$. At $x = 0.1$, $\cos x - 1 = -0.0050$, while $x = 0.1$. The numerator is very small compared with the denominator, so their ratio approaches 0.

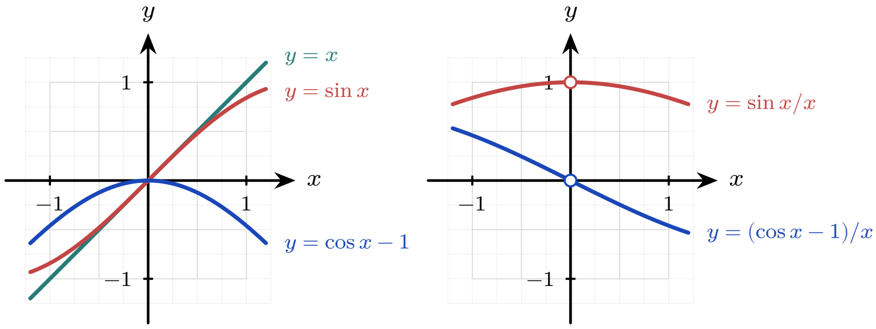

Figure 13.2 **Left:** near the origin $\sin x$ is almost the same as x, while $\cos x - 1$ is almost flat. **Right:** the two quotients. Each has a hole at $x = 0$, but from both sides $\frac{\sin x}{x}$ approaches 1 and $\frac{\cos x - 1}{x}$ approaches 0.

Both limits are needed later: they are the two facts from which the derivatives of $\sin x$ and $\cos x$ are built.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

Not every limit exists. A limit **converges** when the values approach one finite number, and **diverges** when they do not. Three examples show the difference: $\frac{1}{x^2}$ increases without bound as $x \to 0$; $\frac{1}{x}$ increases without bound as $x \to 0^+$ but decreases without bound as $x \to 0^-$; while $\sin x$ oscillates between -1 and 1 as $x \to \infty$ and does not approach any single value. For $\frac{1}{x}$, writing that the limit "equals infinity" hides the disagreement between the two sides, so it is not an acceptable answer. You have met this distinction before: the infinite geometric series of Ch. 2 converges exactly when $|r| < 1$, because its partial sums approach a single value, and diverges otherwise.

Deciding which of the two happens as $x \to \infty$ needs a method, because there is no value of x to substitute. For a ratio of polynomials this method works: divide every term, above and below, by the highest power of x that appears. What is left is a set of constants together with terms like $\dfrac{1}{x}$ and $\dfrac{1}{x^2}$, and each of those approaches 0.

Worked through. To evaluate $\displaystyle\lim_{x \to \infty} \frac{5x - 2}{x}$ and $\displaystyle\lim_{x \to \infty} \frac{4x^2 - x}{3x^2 + 5}$, divide above and below by the highest power of x in each. In the first, the highest power is x, so divide both terms of the numerator by it:

$$\lim_{x\to\infty} \frac{5x-2}{x} = \lim_{x\to\infty} \left(5 - \frac{2}{x}\right) = 5 - 0 = 5.$$

In the second, the highest power is x^2, so divide all four terms by x^2:

$$\lim_{x\to\infty} \frac{4x^2 - x}{3x^2+5} = \lim_{x\to\infty} \frac{4 - \dfrac{1}{x}}{3 + \dfrac{5}{x^2}} = \frac{4-0}{3+0} = \frac{4}{3}.$$

Every term that approaches 0 came from a lower power of x than the one you divided by. When the top and the bottom have the same highest power, as here, what remains is the ratio of the leading coefficients: $\frac{5}{1}$ in the first and $\frac{4}{3}$ in the second.

YOUR TURN 13.1 Limits

Direct substitution and factorising

1. Find each limit by direct substitution.

(a) $\lim_{x\to 2}(3x-1)$

(b) $\lim_{x\to 0}(x^2+2x+5)$

(c) $\lim_{x\to 3}(x^2-9)$

(d) $\lim_{x\to 1}\dfrac{x+1}{x+3}$

2. Each of these gives $\frac{0}{0}$. Factorise, cancel, then take the limit.

(a) $\lim_{x\to 4}\dfrac{x^2-16}{x-4}$

(b) $\lim_{x\to 1}\dfrac{x^2-1}{x-1}$

(c) $\lim_{x\to 2}\dfrac{x^2-x-2}{x-2}$

(d) $\lim_{x\to -3}\dfrac{x^2+x-6}{x+3}$

One-sided limits

3. Explain why $\lim_{x\to 0}\dfrac{|x|}{x}$ does not exist.

4. For each function, find the left-hand and right-hand limits at the value of x given, and state whether the limit exists there.

(a) $f(x) = \begin{cases} x^2, & x < 1 \\ 2x+1, & x \geq 1 \end{cases}$ at $x = 1$

(b) $g(x) = \begin{cases} x+2, & x < 3 \\ 8-x, & x > 3 \end{cases}$ with $g(3) = 1$, at $x = 3$. Compare the limit with $g(3)$.

5. Find $\lim_{x\to 1^-}\dfrac{x^2-1}{|x-1|}$ and $\lim_{x\to 1^+}\dfrac{x^2-1}{|x-1|}$, and hence state whether $\lim_{x\to 1}\dfrac{x^2-1}{|x-1|}$ exists. *(Hint: $|x-1| = 1-x$ when $x < 1$)*

Limits at infinity

6. State whether each of the following converges or diverges as $x \to \infty$, and give the limit where one exists.

(a) $\dfrac{1}{x}$

(b) x^2

(c) e^{-x}

(d) $2 + \dfrac{5}{x}$

7. Evaluate each limit by dividing every term by the highest power of x.

(a) $\lim_{x\to\infty} \dfrac{6x-1}{2x+3}$

(b) $\lim_{x\to\infty} \dfrac{2x^2+3x}{5x^2-1}$

(c) $\lim_{x\to\infty} \dfrac{x+4}{x^2+1}$

Estimating a limit

8. GDC Consider $y = \dfrac{\sin x}{x}$.

(a) Write down the value of $\lim_{x\to 0} \dfrac{\sin x}{x}$.

(b) Graph the curve for $-10 \le x \le 10$ on your GDC, and sketch what you see.

(c) Describe what the graph does at $x = 0$, and what it does as $x \to \pm\infty$.

9. GDC Work in radians.

(a) Evaluate $\dfrac{\cos x - 1}{x}$ for $x = \pm 0.1$ and $x = \pm 0.01$. State the value that its limit as $x \to 0$ appears to take.

(b) Estimate $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$ by evaluating the expression at $x = 0.1$ and $x = 0.01$. State the exact value you think the limit takes.

Concept check

10. A student is asked for $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$ and writes: "*substituting* $x = 3$ *gives* $\frac{0}{0}$, *so the limit does not exist.*" Explain why this conclusion is wrong, and find the limit.

13.2 The Derivative

Calculus is the mathematics of change. You already know how to measure change: it is the gradient.

Rise over run gives the change in y for each unit of change in x. If the gradient doubles, the quantity changes twice as fast.

There is one problem. A gradient is defined for a *straight line* only. One number describes the whole line, because a line changes at the same rate everywhere.

Very few real quantities behave like that. A cooling cup, a thrown ball, a growing population: in each case the rate of change is itself changing. So the question "what is the gradient of this curve?" has no single answer. The answer is different at every point.

So ask a smaller question. Do not ask for the gradient of the curve. Ask for the gradient of the curve *at one point*. Once you can answer that for one point, you can answer it for every point, and the rate of change becomes a function of its own.

Here is the method. On the graph of $y = f(x)$, choose your point x and a second point $x + h$ close to it. Join the two points with a straight line, called the **secant**, and find its gradient. You can already calculate that:

$$\frac{f(x+h) - f(x)}{h},$$

the rise over the run. This is the average rate of change over the interval of width h.

The secant gradient is still not the answer. It is an average over an interval of width h, and the question was about a single point.

Now make h smaller and smaller. The second point moves towards the first, and the secant turns towards one limiting line, which touches the curve at the chosen point. That line is the **tangent**. Its gradient is the rate of change at a single point, and it is called the **derivative**.

That is the idea. A gradient is defined only for a line, so we find the line that matches the curve at that point, and use its gradient instead.

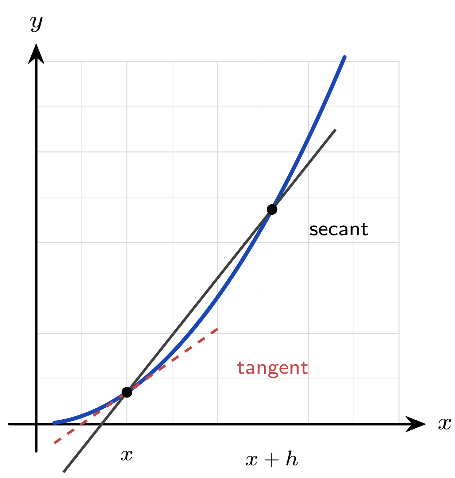

Figure 13.3 The secant joins two points on the curve and measures an average rate of change. As $h \to 0$ the second point moves to the first and the secant approaches the tangent.

> DEF In words: the **derivative** of f at a point is the **gradient of the tangent** to the curve $y = f(x)$ at that point. It is the limit of the gradients of the secants through that point, as the second point moves towards it.

DEF · IB HL The **derivative** of f at x, written $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

when this limit exists. Computing a derivative directly from this definition is called differentiation **from first principles**.

The derivative is itself a function, called the **gradient function**. For each value of x, the number $f'(x)$ is the gradient of the curve $y = f(x)$ at the point with that x-coordinate. For example, the derivative of $f(x) = x^2$ is $f'(x) = 2x$ (§13.3). So the gradient of $y = x^2$ is 2 at $x = 1$, 0 at $x = 0$ and -4 at $x = -2$. One formula, $2x$, gives the gradient at every point of the curve.

13.2.1 Notation, and where it came from

Calculus was developed independently by two people, and each was trying to solve a different problem.

- **Isaac Newton**, in England in the 1660s, was studying *motion*: falling bodies, and later the orbits of the planets under gravity. He needed the speed of a moving object at a single instant, which is the problem of the speedometer at the start of this chapter.
- **Gottfried Leibniz**, in Germany and France in the 1670s, was studying the *geometry of curves*: how to find the tangent to any curve, and the area under it. He wanted one set of symbols and rules that would work for every curve.

Both built on earlier methods for tangents by Pierre de Fermat and Isaac Barrow. Newton's own notation, a dot above the variable, is still seen in physics, but it is not used in this course. The notation used here comes from Leibniz.

The IB course uses these forms for the first derivative:

$$\frac{dy}{dx}, \qquad f'(x), \qquad \frac{dV}{dr}, \qquad \frac{ds}{dt}.$$

Leibniz wrote $\frac{dy}{dx}$, built from $\frac{\Delta y}{\Delta x}$ for a finite change, with the d marking the limit as the change approaches zero. The form $f'(x)$ names the function being differentiated. The form $\frac{d}{dx}f(x)$, used in the formula cards of this chapter, means "the derivative of $f(x)$ with respect to x".

The Leibniz form is the one that carries the most information, because it names both variables. When the quantities are not called x and y, the letters change with them. A volume V changing with radius r has derivative $\frac{dV}{dr}$, and a displacement s changing with time t has derivative $\frac{ds}{dt}$. Read each as a rate: how fast the top quantity changes per unit of the bottom one. Examination questions use these forms often, and choosing the right letters is an important part of setting up the problem.

One caution about $\dfrac{dy}{dx}$. It is written as a fraction, but it is not one. It is a single symbol standing for the limit of the fraction $\dfrac{\Delta y}{\Delta x}$, and dy and dx have no independent meaning at this level. The notation is still well designed, because the limit keeps some of the behaviour of a fraction. That is why the chain rule in §13.6 can be written as $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$, with the du appearing to cancel. The cancelling is a memory aid, not a proof, but it works because the underlying limits behave that way.

Leibniz's notation became the standard one, largely for a practical reason: it shows the variable of differentiation, so it extends directly to the chain rule, to implicit differentiation, and to integration, where $\int f(x)\,dx$ carries the same dx.

13.2.2 Where derivatives are used

A derivative is a rate of change, and rates of change appear wherever one quantity depends on another.

- **Motion.** Differentiate displacement with respect to time to get velocity, and again to get acceleration. This is how a vehicle's braking distance is calculated, and it is the subject of Ch. 14.
- **Optimisation.** A curve is flat where its derivative is zero, so setting $f'(x) = 0$ locates maxima and minima. Manufacturers use this to find the packaging that uses the least material for a required volume.
- **Medicine.** The rate at which a drug leaves the bloodstream determines the dosing interval, and that rate is a derivative of the concentration with respect to time.
- **Economics.** *Marginal* cost is the derivative of total cost with respect to quantity produced, and it answers what one more unit costs to make.
- **Computer graphics.** A surface is shaded according to the direction it faces, which is found from derivatives of the surface, and this shading helps rendered objects look solid.

Every one of these is the same calculation, applied to a different pair of quantities.

To differentiate from first principles, substitute into $\dfrac{f(x+h) - f(x)}{h}$, expand, and simplify until the factor h in the denominator cancels. Only then let $h \to 0$, because the division by h is valid only while $h \neq 0$.

Worked through. To differentiate $f(x) = x^2 + 4x$ from first principles, form the difference quotient and expand:

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 + 4(x+h) - x^2 - 4x}{h} = \frac{2xh + h^2 + 4h}{h} = 2x + h + 4.$$

Now let $h \to 0$:

$$f'(x) = \lim_{h \to 0} (2x + h + 4) = 2x + 4.$$

You must cancel the h before taking the limit. While $h \neq 0$ the division is allowed. Only after cancelling is it safe to set h to zero.

EXAMPLE 13.1

Differentiate $f(x) = \dfrac{1}{x}$ from first principles.

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \frac{1}{h}\cdot\frac{x-(x+h)}{x(x+h)} = \frac{-h}{h\,x(x+h)} = \frac{-1}{x(x+h)}.$$

As $h \to 0$:

$$f'(x) = \lim_{h\to 0}\frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$

TIP In examinations, first-principles differentiation questions usually involve polynomials. The $\frac{1}{x}$ example above shows how far the definition extends; it is not itself an examination task.

13.2.3 Continuity and differentiability HL

The derivative is defined as a limit, and a limit can fail to exist. So a function need not have a derivative at every point. Two ideas describe where it does.

A function f is **continuous** at $x = a$ when three things are true:

1. $f(a)$ is defined;
2. $\lim\limits_{x\to a} f(x)$ exists, so the left-hand and right-hand limits are equal (§13.1);
3. this limit equals $f(a)$.

On a graph, a continuous function has no hole and no jump at that point: you can draw through it without lifting the pen.

A function f is **differentiable** at $x = a$ when the first-principles limit $f'(a)$ exists. On a graph, the curve is continuous there and also *smooth*: it has one non-vertical tangent at that point, with no corner.

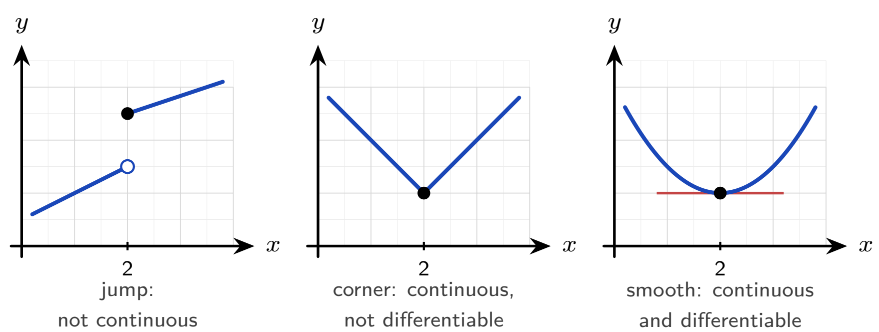

Figure 13.4 Three behaviours at $x = 2$. An open circle marks a point that is not on the graph. Only the smooth curve on the right has a single tangent (scarlet) at the point.

The two ideas are linked in one direction only.

- If f is differentiable at a point, then it is continuous there. A jump or a hole always prevents a derivative.

- If f is continuous at a point, it may still fail to be differentiable there. A corner is the usual reason.

The absolute value function shows the second case.

Worked through. To show that $f(x) = |x-1|$ is continuous at $x = 1$ but not differentiable there, check the two properties in turn. For continuity: $f(1) = 0$, and as $x \to 1$ from either side, $|x-1| \to 0$. The limit exists and equals $f(1)$, so f is continuous at $x = 1$.

For differentiability, form the difference quotient at 1:

$$\frac{f(1+h)-f(1)}{h} = \frac{|h|}{h} = \begin{cases} 1 & h > 0, \\ -1 & h < 0. \end{cases}$$

The quotient approaches 1 from the right and -1 from the left. The one-sided limits are not equal, so the limit does not exist and f has no derivative at $x = 1$. On the graph, the V has a corner there: the gradient is -1 just to the left and 1 just to the right, so there is no single tangent.

So $|x-1|$ is continuous at every real number, but it is not differentiable at $x = 1$.

TIP Examinations focus on the idea rather than on formal tests: you may differentiate only where the curve is smooth.

YOUR TURN 13.2 The Derivative

Notation

11.

(a) Write down three notations for the derivative of $y = f(x)$.

(b) A volume V depends on a radius r. Write down the notation for the rate of change of V with respect to r, and state in words what it measures.

First principles

12. Differentiate each from first principles, showing the limit.

(a) $f(x) = 3x^2$

(b) $f(x) = x^2 - 3x$

(c) $f(x) = 5 - 2x^2$

13. Differentiate $f(x) = x^3$ from first principles. *(Hint: $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$)*

14. Differentiate $f(x) = \dfrac{2}{x}$ from first principles.

Estimating a gradient numerically

15. GDC The definition of the derivative can also be used numerically. For $f(x) = \sin x$, evaluate $\dfrac{f(0+h) - f(0)}{h}$ for $h = 0.1$, $h = 0.01$ and $h = 0.001$, working in radians. State the value the quotient is approaching, and what this suggests about the gradient of $y = \sin x$ at $x = 0$.

16. GDC Repeat the same calculation for $f(x) = e^x$ at $x = 0$, then at $x = 1$. Explain what the two answers suggest about the derivative of e^x.

Continuity and differentiability

17. For each function, state which of the three conditions for continuity at $x = 2$ fails, if any.

(a) $f(x) = \dfrac{x^2 - 4}{x - 2}$

(b) $f(x) = \begin{cases} x+1, & x < 2 \\ x^2, & x \geq 2 \end{cases}$

(c) $f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$

18. Let $g(x) = \begin{cases} x+1, & x < 1 \\ 3-x, & x \geq 1. \end{cases}$

(a) Show that g is continuous at $x = 1$.

(b) Use the difference quotient at $x = 1$ to show that g is not differentiable at $x = 1$.

(c) A function h has a jump at $x = 4$. Explain why h cannot be differentiable at $x = 4$.

Concept check

19. A student differentiates $f(x) = x^2$ from first principles and writes $\dfrac{(x+h)^2 - x^2}{h} = \dfrac{x^2 + h^2 - x^2}{h} = h$, which tends to 0 as $h \to 0$, so $f'(x) = 0$. Identify the error and correct the working.

13.3 Differentiating Powers

First principles works, but it is slow. The two first-principles calculations above show a pattern: the x^2 term gave $2x$, and $\frac{1}{x} = x^{-1}$ gave $-x^{-2}$.

In both cases two things happened. The power decreased by one, and the original power became a multiplier in front. This is the **power rule**.

KEY · IB

$$\frac{d}{dx}x^n = nx^{n-1}$$

The Standard Level syllabus states this for integer powers in SL 5.3 and extends it to rational powers in SL 5.6. The rational form is what allows $\sqrt{x}$ to be differentiated below, and it is needed throughout §13.5 onwards.

The rule applies only to a term written as x^n, so rewrite every root and every reciprocal as a power first: $\sqrt{x} = x^{1/2}$ and $\frac{1}{x^2} = x^{-2}$.

Two more rules let you differentiate any polynomial. A constant multiplier is unchanged, and a sum is differentiated one term at a time:

$$\frac{d}{dx}(c\,f(x)) = c\,f'(x), \qquad \frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x).$$

A constant on its own has derivative zero, because its graph is a horizontal line with gradient zero.

Worked through. To differentiate $y = 3x^4 - 2x^2 + 7$, apply the power rule to each term:

$$\frac{dy}{dx} = 3(4x^3) - 2(2x) + 0 = 12x^3 - 4x.$$

The constant 7 has derivative zero, as every constant does.

EXAMPLE 13.2

Differentiate $y = \sqrt{x} + \dfrac{1}{x^2}$.

Rewrite both terms as powers before differentiating:

$$y = x^{1/2} + x^{-2} \implies \frac{dy}{dx} = \tfrac{1}{2}x^{-1/2} - 2x^{-3} = \frac{1}{2\sqrt{x}} - \frac{2}{x^3}.$$

The power rule requires the function in the form x^n. Convert every root and every reciprocal to a power before differentiating, and the rest of the question becomes routine.

EXAMPLE 13.3

Find the gradient of $y = x^3 - 5x$ at the point where $x = 2$.

Differentiate, then substitute:

$$\frac{dy}{dx} = 3x^2 - 5, \qquad \left.\frac{dy}{dx}\right|_{x=2} = 3(4) - 5 = 7.$$

The derivative is a function. A gradient *at a point* is that function evaluated at the point.

YOUR TURN 13.3 Differentiating Powers

Polynomials

20. Differentiate each of the following.

(a) $y = x^7$
(b) $y = 5x^3 - 2x$
(c) $y = x^4 - 3x^2 + 6$
(d) $y = 4x^5 + x$
(e) $y = 2x^3 - 7x^2 + x - 9$
(f) $y = 6$

Rewriting as a power

21. Rewrite each function as a power of x, then differentiate. In part (f), divide through first.

(a) $y = 2/x$
(b) $y = \sqrt{x}$
(c) $y = x^3 + 1/x$
(d) $y = 1/x^3$
(e) $y = \sqrt[3]{x}$
(f) $y = (x^2 + 1)/x$

22. Expand or divide to write each function as a sum of powers of x, then differentiate.

(a) $y = (2x + 1)^2$
(b) $y = \dfrac{x^3 - 4}{x^2}$
(c) $y = \dfrac{x + 1}{\sqrt{x}}$

Gradient at a point

23. Find the gradient of each curve at the point given.

(a) $y = x^2 - 5x$ at $x = 3$
(b) $y = x^3 - 4x$ at $x = -1$
(c) $y = \sqrt{x}$ at $x = 4$
(d) $y = 2x^3 + x$ at $x = 1$

24. Find the value or values of x at which each curve has the stated gradient.

(a) $y = x^2 - 6x$, gradient 0
(b) $y = x^3$, gradient 12
(c) $y = x^3 - 3x^2 + 1$, gradient 9

Concept check

25. A student differentiates $y = \dfrac{3}{x^2}$ and writes $y = 3x^{-2}$, so $\dfrac{dy}{dx} = -6x^{-1}$. Identify the error and give the correct derivative.

13.4 Tangents and Normals

The derivative gives the gradient of the tangent at any point, and a gradient plus a point determines a line. So you can now write the equation of the **tangent** to a curve at a given point. The **normal** is the line perpendicular to the tangent there, with gradient the negative reciprocal.

The tangent and the normal are both straight lines, so both are written with the same equation, the point-gradient form from Ch. 4:

$$y - y_1 = m(x - x_1),$$

where (x_1, y_1) is a known point on the line and m is its gradient. The point (x_1, y_1) is the same for both lines, because the tangent and the normal meet the curve at the same place. Only the gradient m is different.

KEY At the point (x_1, y_1) on the curve $y = f(x)$, both lines have the form $y - y_1 = m(x - x_1)$, with

$$m_{\text{tangent}} = f'(x_1), \qquad m_{\text{normal}} = -\frac{1}{f'(x_1)}$$

(for $f'(x_1) \neq 0$; if $f'(x_1) = 0$ the normal is the vertical line $x = x_1$).

So a question of this type follows three steps: find y_1 by substituting x_1 into f, find $f'(x_1)$ for the gradient, then substitute both into the point-gradient form.

On a calculator paper the same job can be done from the screen. A graphic display calculator will evaluate a numerical derivative at a point, and most will draw the tangent there and report its equation. Two cautions apply. Many calculators give only the *tangent*, so a normal still needs the $-1/f'(x_1)$ step by hand. And a numerical answer earns few marks on its own: show the gradient you used and the point you used it at, then quote the line. Use the screen to check the algebra, not to replace it.

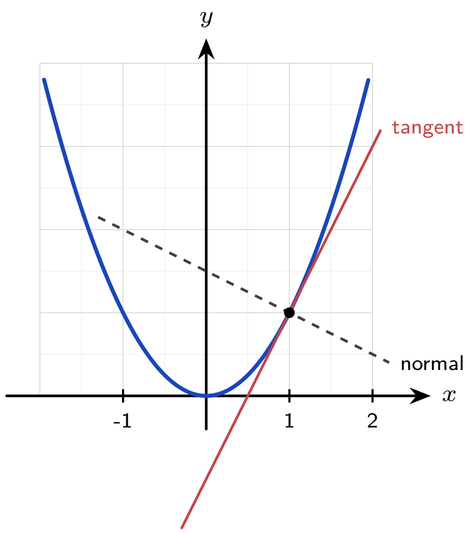

Figure 13.5 At a point on a curve the tangent has gradient $f'(x)$, and the normal is perpendicular to it, with gradient $-1/f'(x)$.

Worked through. To find the tangent and normal to $y = x^2 - 3x + 2$ at the point where $x = 2$, first find the point: $y(2) = 4 - 6 + 2 = 0$, so the point is $(2, 0)$. Then find the gradient: $\dfrac{dy}{dx} = 2x - 3$, so at $x = 2$ the tangent gradient is $2(2) - 3 = 1$. The tangent passes through $(2, 0)$ with gradient 1:

$$y - 0 = 1(x - 2) \implies y = x - 2.$$

The normal is perpendicular, so its gradient is -1:

$$y - 0 = -1(x - 2) \implies y = -x + 2.$$

Always find the y-coordinate first. A common error is to write the equation of the line using the x-value alone.

13.4.1 Tangents through a point off the curve

A harder question names a point the tangent must pass *through*, and that point is not on the curve. The point where the tangent touches the curve, the **point of contact**, is not given, so make it the unknown.

1. Let the point of contact be $(a, f(a))$.
2. Write the equation of the tangent there: $y - f(a) = f'(a)(x - a)$.
3. Substitute the coordinates of the given point, because the tangent passes through it. This gives an equation in a alone.
4. Solve for a, and substitute each value back into the tangent equation.

For a quadratic curve the equation in a is quadratic, so it usually has two roots, and each root gives a separate tangent.

Worked through. To find the equations of both tangents to $y = x^2 - 2x + 3$ that pass through the point $(2, -1)$, first check where the point sits: at $x = 2$ the curve is at $4 - 4 + 3 = 3$, not -1, so $(2, -1)$ is off the curve and is not the point of contact.

Let the contact point be $(a,\, a^2 - 2a + 3)$. Since $\dfrac{dy}{dx} = 2x - 2$, the tangent there has gradient $2a - 2$, so

$$y - (a^2 - 2a + 3) = (2a - 2)(x - a) \implies y = (2a - 2)x - a^2 + 3.$$

Now substitute the point $(2, -1)$, because the tangent passes through it:

$$-1 = 2(2a - 2) - a^2 + 3 \implies a^2 - 4a = 0 \implies a(a - 4) = 0,$$

so $a = 0$ or $a = 4$. Substituting each value back into $y = (2a - 2)x - a^2 + 3$:

$$a = 0: \quad y = -2x + 3, \qquad\qquad a = 4: \quad y = 6x - 13.$$

Two roots give two tangents, and both pass the check at the given point: $-2(2) + 3 = -1$ and $6(2) - 13 = -1$.

YOUR TURN 13.4 Tangents and Normals

Tangents

26. Find the equation of the tangent to each curve at the point given.

(a) $y = x^2$ at $(2, 4)$

(b) $y = x^2 - 4x + 3$ at $x = 1$

(c) $y = 1/x$ at $x = 1$

(d) $y = \sqrt{x}$ at $x = 9$

27. Find the coordinates of the point on $y = x^2 - 6x$ at which the tangent is horizontal.

Normals

28. Find the equation of the normal to each curve at the point given.

(a) $y = x^3$ at $(1, 1)$

(b) $y = \sqrt{x}$ at $x = 4$

(c) $y = \dfrac{2}{x}$ at $x = 1$

29.

(a) A tangent and a normal are drawn at the same point of a curve. Explain the relationship between their gradients.

(b) Find the equation of the normal to $y = x^2 - 4x + 7$ at the point where $x = 2$.

Tangents through a point off the curve

30. The tangent to $y = x^2$ at the point where $x = a$ passes through $(0, -4)$.

(a) Find the two possible values of a.

(b) Write down the two tangents, and check that each one passes through $(0, -4)$.

31. Find the equations of the two tangents to $y = x^2 + 3$ that pass through the point $(1, 0)$.

Concept check

32. A student finds the tangent to $y = x^2$ at the point $(3, 9)$ and writes $y - 9 = 2x(x - 3)$. Identify the error and find the correct equation of the tangent.

13.5 The Standard Derivatives

The power rule covers algebraic functions. The trigonometric, exponential and logarithmic functions have derivatives of their own. Each can be proved from first principles. You can also find each one by reading gradients from a graph. Do that once before memorising anything.

The idea is to move along the graph of f from left to right, and at each point plot its gradient as a height. The curve you plot is f'. In practice, sketch it in this order.

1. **Start with the peaks and troughs of f.** There the tangent is horizontal, so $f'(x) = 0$. Mark these x-values on the axis: the graph of f' meets the x-axis there.
2. **Where f is increasing**, its gradient is positive, so draw f' *above* the x-axis.
3. **Where f is decreasing**, its gradient is negative, so draw f' *below* the x-axis.
4. **Where f is steepest**, f' is furthest from the axis.

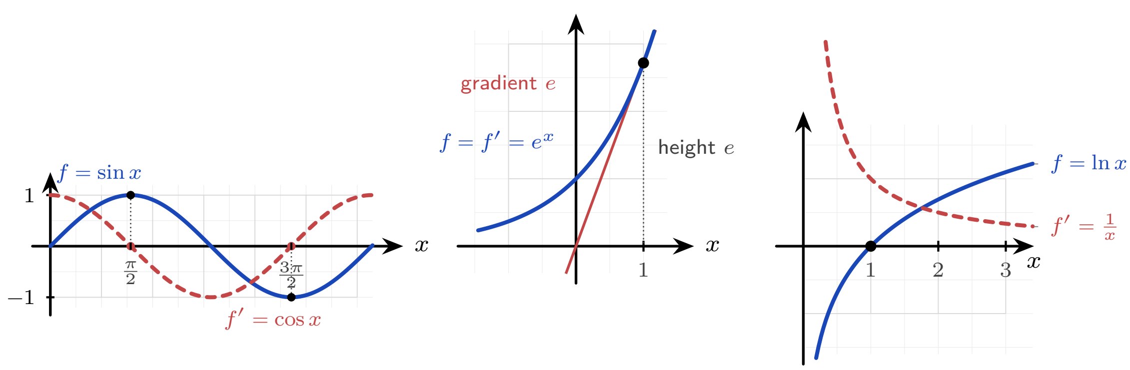

Figure 13.6 Each derivative read from the gradient of its own graph. **Left:** where $\sin x$ is flat, at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, the curve $\cos x$ crosses zero. **Centre:** the tangent to e^x at $x = 1$ has gradient e, which is also the height of the curve there. **Right:** $\ln x$ is steep near 0 and flattens as x grows, and $\frac{1}{x}$ does exactly that.

Read each panel separately.

Sine. At $x = 0$ the sine graph is increasing most steeply, so f' starts at its maximum. At the top of the curve, $x = \frac{\pi}{2}$, the tangent is horizontal, so f' is zero. After the top the graph decreases, so f' is negative. A curve that starts at 1, decreases through zero at $\frac{\pi}{2}$ and reaches -1 at π is exactly $\cos x$.

The exponential. Measure the gradient of $y = e^x$ at any point and you get the height of the curve at that point. That is the defining property of e, so here $f' = f$.

The logarithm. Just to the right of 0 the graph of $\ln x$ is almost vertical, so f' is very large. As x increases the curve becomes flatter, so f' decreases towards zero. It stays positive everywhere. That describes $\dfrac{1}{x}$.

Those are the four standard derivatives to learn.

KEY · IB

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x, \quad \frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}\ln x = \frac{1}{x}$$

Note the exponential in particular: e^x is its own derivative. More precisely, the functions equal to their own derivative are exactly those of the form $y = Ce^x$, and e^x is the one passing through $(0, 1)$. That property is one reason e matters so much in calculus. Note also that the results for $\sin$ and $\cos$ hold only when x is measured in *radians*, and there is a reason for that. The derivative of $\sin x$ is built on the limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$ from §13.1, and that limit equals 1 only in radians. Measured in degrees the same limit is $\frac{\pi}{180}$, so the derivative would come out as $\frac{\pi}{180}\cos x$ instead. Radians are simply the unit that makes the constant equal to 1, which is why calculus uses them throughout.

At Higher Level a wider set of standard derivatives is provided. HL They are listed here for reference and used freely in the rules that follow. The reciprocal and inverse trigonometric functions in the table are defined in Ch. 8; only their derivatives are needed here.

KEY · IB HL

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x \qquad \frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}a^x = a^x \ln a \qquad \frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Notice one pair. The derivative of $\arccos$ is the negative of the derivative of $\arcsin$, because the two graphs are reflections of each other.

Worked through. To differentiate $y = 3\sin x + 2e^x - 5\ln x$, apply the standard derivatives term by term:

$$\frac{dy}{dx} = 3\cos x + 2e^x - \frac{5}{x}.$$

Constant multipliers are unchanged, exactly as with powers.

The Higher Level table is used the same way. Read each derivative off it, keep the constant multipliers in place, and differentiate one term at a time.

EXAMPLE 13.4

Differentiate (a) $y = 3\tan x - 2\cos x$, (b) $y = 5^x + \ln x$, (c) $y = \arccos x + 3\arctan x$.

(a)

$$\frac{dy}{dx} = 3\sec^2 x + 2\sin x.$$

The derivative of $\cos x$ is $-\sin x$, so subtracting a cosine adds a sine.

(b)

$$\frac{dy}{dx} = 5^x \ln 5 + \frac{1}{x}.$$

The factor $\ln 5$ is a constant, and it appears because $5^x = e^{x\ln 5}$. For e^x the factor is $\ln e = 1$, which is why e^x has no extra factor.

(c)

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} + \frac{3}{1+x^2}.$$

The leading minus sign is the only thing separating the $\arccos$ row of the table from the $\arcsin$ row. Neither inverse trigonometric derivative contains a trigonometric function; both are algebraic, which is what makes them the results you meet again as integrals in Ch. 15.

YOUR TURN 13.5 The Standard Derivatives

The four standard derivatives

33. Differentiate each of the following.

(a) $y = \sin x + \cos x$

(b) $y = e^x - 3\ln x$

(c) $y = 4\cos x$

(d) $y = 2\sin x - 5e^x$

34. The results $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$ hold only when x is measured in radians. Explain why the choice of unit matters.

The Higher Level table

35. Differentiate each of the following.

(a) $y = \tan x$ (b) $y = \arctan x$
(c) $y = 3^x$ (d) $y = \ln x + \tan x$
(e) $y = \sec x$ (f) $y = \arcsin x$

36. Differentiate each of the following.

(a) $y = \operatorname{cosec} x$ (b) $y = 2\cot x$
(c) $y = \log_2 x$ (d) $y = 3\arccos x - 4^x$

Gradients and tangents

37. Find the gradient of $y = \sin x$ at $x = \dfrac{\pi}{3}$, giving an exact answer.

38. Find the point on the curve $y = e^x$ at which the gradient is 1, and write down the equation of the tangent there.

39. Find the equation of the tangent to $y = \ln x$ at the point where $x = e$.

40. Find the values of x, for $0 \le x \le 2\pi$, at which the curve $y = x + 2\cos x$ has gradient zero.

Concept check

41. A student writes $\dfrac{d}{dx}2^x = x \cdot 2^{x-1}$. Explain why the power rule does not apply here, and give the correct derivative.

13.6 The Rules of Differentiation

Most functions are built from simpler ones, by composing, multiplying or dividing them. Three rules cover these three cases. With the standard derivatives, they let you differentiate almost any function in the course.

13.6.1 The chain rule

A **composite** function is a function inside a function, such as $(2x^2+3)^5$ or $\sin(2x)$. The **chain rule** differentiates it by working from the outside in: differentiate the outer function, then multiply by the derivative of the inner.

KEY · IB If $y = g(u)$ where $u = f(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

The meaning of the rule is that **rates multiply**. Suppose a car's fuel use y depends on its speed u, and its speed depends on time x. If fuel use rises 3 units for every unit of speed, and speed rises 2 units for every unit of time, then fuel use rises $3 \times 2 = 6$ units for every unit of time. That is exactly $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. A chain of dependencies multiplies its rates, in the same way that gears in a machine multiply their turning ratios.

The power rule is itself a case of the chain rule. In $y = x^7$ the inner function is just x, so the chain rule gives

$$\frac{dy}{dx} = 7x^6 \times \frac{d}{dx}(x) = 7x^6 \times 1 = 7x^6.$$

The factor 1 is the derivative of the inner function x, and it is not usually written. Replace the inner function x by $2x + 1$ and the same rule gives

$$\frac{d}{dx}(2x+1)^7 = 7(2x+1)^6 \times 2 = 14(2x+1)^6.$$

Now the derivative of the inner function is 2, not 1, so it must be written.

The chain rule is needed whenever the function is a composite. Two habits make it reliable:

1. **Identify the inner function first**, before differentiating anything. Ask which function is applied to x first: that one is the inner function.
2. **Multiply by the derivative of the inner function.** Differentiating the outer function alone and stopping there is a common error, so check for it at the end of every chain rule question.

The outer function need not be a power. Any of the standard functions of §13.5, such as e^u, $\ln u$ or $\sin u$, can be the outer function, and the rule does not change: differentiate the outer function, keep the inner function unchanged inside it, then multiply by the derivative of the inner function.

EXAMPLE 13.5

Differentiate (a) $y = \ln(2 + \cos x)$ and (b) $y = \sqrt{x^3 + 2}$.

(a) Outer is $\ln u$, inner is $u = 2 + \cos x$. The outer derivative is $\frac{1}{u}$, which must be written at $u = 2 + \cos x$:

$$\frac{dy}{dx} = \frac{1}{2 + \cos x} \cdot (-\sin x) = -\frac{\sin x}{2 + \cos x}.$$

(b) Rewrite the root as a power first, $y = (x^3 + 2)^{1/2}$, so the outer is $u^{1/2}$ and the inner is

$u = x^3 + 2$:

$$\frac{dy}{dx} = \tfrac{1}{2}(x^3+2)^{-1/2} \cdot 3x^2 = \frac{3x^2}{2\sqrt{x^3+2}}.$$

The root itself needs $x^3 + 2 \geq 0$, and the derivative puts that root in a denominator, so this expression holds for $x^3 + 2 > 0$, that is $x > -\sqrt[3]{2}$.

Part (a) shows a common error to avoid: the outer derivative of a logarithm is $\dfrac{1}{\text{inner}}$, so it is $\dfrac{1}{2+\cos x}$ here, never $\dfrac{1}{x}$.

13.6.2 The product rule

To differentiate a product of two functions, you cannot simply multiply their derivatives. The mistake is a natural one to make, and the reason it fails explains the correct rule.

Take $u = x$ and $v = x$, so that $y = uv = x^2$. The derivative of x^2 is $2x$. But $u' = 1$ and $v' = 1$, and $u'v' = 1$. The two answers do not agree, so multiplying the derivatives gives a wrong answer.

The reason is that a product changes for two separate reasons at the same time.

Suppose both u and v increase. Part of the increase in uv comes from the change in u, multiplied by the whole of v. The rest comes from the change in v, multiplied by the whole of u. The correct rule adds these two parts, which is why each term keeps one factor un-differentiated.

A rectangle makes this concrete. Let u and v be the two side lengths, so that uv is the area. Now stretch both sides slightly. The extra area has three parts. There is a thin strip along the top, of area (change in v) $\times\, u$, and a thin strip along the side, of area (change in u) $\times\, v$. There is also a tiny corner rectangle where the two strips meet. That corner is a product of two small quantities, so its area approaches zero faster than the strips and adds nothing in the limit. The two strips are the two terms of the product rule.

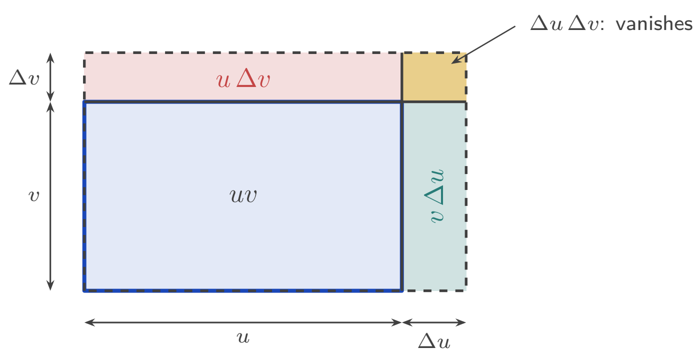

Figure 13.7 The product rule as an area. Growing both sides of a rectangle adds a strip along the top of area $u\,\Delta v$, a strip along the side of area $v\,\Delta u$, and a small corner of area $\Delta u\,\Delta v$. The two strips give the two terms of the rule; the corner is a product of two small quantities and vanishes in the limit.

KEY · IB If $y = uv$, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Worked through. To differentiate $y = (x^2+1)e^x$, let $u = x^2+1$ and $v = e^x$, so $u' = 2x$ and $v' = e^x$:

$$\frac{dy}{dx} = (x^2+1)e^x + e^x(2x) = e^x(x^2+2x+1) = (x+1)^2e^x.$$

Factorise the answer where you can. A factorised form makes later work much easier, for example finding where $\frac{dy}{dx} = 0$.

The rules combine, and many examination questions require it. When one factor of a product is itself a composite, the product rule gives the structure of the answer and the chain rule gives that factor's derivative. Find each of u' and v' completely before assembling anything.

EXAMPLE 13.6

Differentiate $y = x^3e^{2x}$.

Let $u = x^3$ and $v = e^{2x}$, so $u' = 3x^2$. The second factor is a composite, with outer e^w and inner $w = 2x$, so the chain rule gives

$$v' = e^{2x} \cdot 2 = 2e^{2x}.$$

Now assemble the product rule:

$$\frac{dy}{dx} = x^3\left(2e^{2x}\right) + e^{2x}\left(3x^2\right) = e^{2x}\left(2x^3+3x^2\right) = x^2(2x+3)e^{2x}.$$

The exponential factor is never zero, so the factorised form shows at once that the gradient is zero only at $x = 0$ and $x = -\frac{3}{2}$.

13.6.3 The quotient rule

For a quotient of two functions, the rule is similar, but the order of the terms now matters because a subtraction is involved.

The rule is not a new idea. A quotient $\frac{u}{v}$ is the product $u \times v^{-1}$, so the product and chain rules together produce the quotient rule.

The minus sign has a plain meaning. Increasing the top of a fraction increases its value, so the u term is positive. Increasing the bottom *decreases* the value, so the v term is negative. The v^2 in the denominator comes from differentiating v^{-1}: the chain rule gives $-v^{-2}\frac{dv}{dx}$.

KEY · IB If $y = \dfrac{u}{v}$, then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

Worked through. To differentiate $y = \dfrac{x}{x^2+1}$, let $u = x$ and $v = x^2 + 1$, so $u' = 1$ and $v' = 2x$:

$$\frac{dy}{dx} = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

The order in the numerator is: bottom times derivative of top, minus top times derivative of bottom. Reversing the two terms changes the sign of the answer, and it is a common error.

YOUR TURN 13.6 The Rules of Differentiation

The chain rule

42. Use the chain rule to differentiate each of the following.

(a) $y = (2x+1)^4$
(b) $y = \sin(x^2)$
(c) $y = e^{3x}$
(d) $y = \sqrt{3x+1}$
(e) $y = \ln(x^2+1)$
(f) $y = \ln(3x-2)$

43. Find the gradient of $y = (x^2-1)^3$ at the point where $x = 2$.

44. A student writes $\dfrac{d}{dx}\sin(3x) = \cos(3x)$. Identify the error and give the correct derivative.

45. Use the chain rule to differentiate each of the following.

(a) $y = (2x^2+3)^5$
(b) $y = e^{x^2}$

The product rule

46. Use the product rule to differentiate each of the following.

(a) $y = x\cos x$
(b) $y = x^2 e^x$
(c) $y = x^3 \ln x$
(d) $y = x\sin x$

47. Explain, with a counterexample, why the derivative of a product is not the product of the derivatives.

The quotient rule

48. Use the quotient rule to differentiate each of the following.

(a) $y = \dfrac{x}{x+1}$ (b) $y = \dfrac{x+1}{x-1}$

(c) $y = \dfrac{\sin x}{x}$ (d) $y = \dfrac{e^x}{x}$

Combining the rules

49. Each of these is a product in which one factor needs the chain rule. Differentiate, and factorise your answer where possible.

(a) $y = x^2 e^{3x}$

(b) $y = x \sin 2x$

(c) $y = x\sqrt{1+x^2}$

Concept check

50. A student differentiates $y = \dfrac{x^2}{x+1}$ with the quotient rule and writes $\dfrac{dy}{dx} = \dfrac{x^2 \cdot 1 - (x+1) \cdot 2x}{(x+1)^2}$. Identify the error and give the correct derivative.

13.7 Higher Derivatives

A car that is speeding up has a velocity that is changing. To describe how fast that velocity changes, you need the rate of change of a rate of change. The derivative $f'(x)$ is itself a function, so it can be differentiated again.

Differentiating $f'(x)$ gives the **second derivative**. It is written $f''(x)$, or $\dfrac{d^2y}{dx^2}$ when $y = f(x)$.

DEF The **second derivative** of f is the derivative of f':

$$f''(x) = \frac{d}{dx}(f'(x)), \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

The notation $\dfrac{d^2y}{dx^2}$ records the two steps. The operation $\dfrac{d}{dx}$ is applied twice, so the d on top carries a 2 and the dx below is written dx^2. Read it as "d two y by d x two". The 2 counts differentiations. It is not a power of y.

The second derivative measures how fast the gradient changes. For motion along a line this has a familiar name. If s is displacement at time t, then $\dfrac{ds}{dt}$ is velocity and $\dfrac{d^2s}{dt^2}$ is acceleration. See §14.4.

To find a second derivative, differentiate once, simplify, then differentiate the result. Every rule of §13.6 applies at each step, because each step is an ordinary differentiation.

Worked through. To find $f'(x)$, $f''(x)$ and $f''(2)$ for $f(x) = x^3 - 9x^2 + 5$, differentiate once, then differentiate the result:

$$f'(x) = 3x^2 - 18x, \qquad f''(x) = 6x - 18.$$

So $f''(2) = 12 - 18 = -6$. Each differentiation lowers the degree of a polynomial by one, so a cubic has a linear second derivative.

EXAMPLE 13.7

Find $\dfrac{d^2y}{dx^2}$ for $y = xe^{2x}$.

The product rule gives the first derivative:

$$\frac{dy}{dx} = e^{2x} + 2xe^{2x} = (1+2x)e^{2x}.$$

This is again a product, so apply the product rule a second time:

$$\frac{d^2y}{dx^2} = 2e^{2x} + (1+2x)\cdot 2e^{2x} = 4(1+x)e^{2x}.$$

Taking out the common factor e^{2x} after each step keeps the expression short enough to differentiate again.

You can continue past the second derivative. Differentiating n times gives the **nth derivative**, written $\dfrac{d^ny}{dx^n}$ or $f^{(n)}(x)$. The brackets matter: $f^{(3)}(x)$ is the third derivative, while $f^3(x)$ usually means $\big(f(x)\big)^3$. For the first three derivatives, dashes are used instead: f', f'', f'''.

Some functions give derivatives that follow a pattern. Once the pattern is found, it gives every derivative at once.

$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$f^{(n)}(x)$
e^{2x}	$2e^{2x}$	$4e^{2x}$	$8e^{2x}$	$16e^{2x}$	$2^n e^{2x}$
$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	repeats every 4
x^{-1}	$-x^{-2}$	$2x^{-3}$	$-6x^{-4}$	$24x^{-5}$	$(-1)^n\, n!\, x^{-(n+1)}$

Each pattern in the last column is a claim about every n, and proof by induction Ch. 3 is how such a claim is proved. Higher derivatives return in Ch. 16, where the derivatives of a function at one point are used to rebuild the function as a Maclaurin series. What the sign of f'' says about the shape of a graph is developed in Ch. 14.

YOUR TURN 13.7 Higher Derivatives

Second derivatives

51. Find the second derivative of each of the following.

(a) $f(x) = x^3 - 6x^2$

(b) $y = \sqrt{x}$

(c) $y = \cos 2x$

(d) $y = e^{-3x}$

52. Find $\dfrac{d^2y}{dx^2}$ for $y = \ln(2x)$.

53. A particle has displacement $s = t^3 - 6t^2$ metres at time t seconds. Find $\dfrac{d^2s}{dt^2}$ when $t = 3$.

54. In each case the first derivative is again a product, so the product rule is needed twice. Find $\dfrac{d^2y}{dx^2}$.

(a) $y = x^2 e^x$

(b) $y = x \sin x$

Third and higher derivatives

55. Find $f'''(x)$ for $f(x) = x^5 - 2x^3$.

56. For $f(x) = x^4$, find $f^{(3)}(x)$ and $\left(f(x)\right)^3$, and evaluate each at $x = 1$.

57. For $f(x) = e^{-x}$, find $f'(x)$, $f''(x)$ and $f'''(x)$, and hence write down $f^{(n)}(x)$.

58. Let $f(x) = \sin x$.

(a) Find $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$.

(b) Hence write down $f^{(10)}(x)$ and $f^{(23)}(x)$.

Concept check

59. Let $f(x) = x^3 - 2x^2$. A student writes $f'(2) = 12 - 8 = 4$, then differentiates 4 to obtain $f''(2) = 0$. Identify the error and find the correct value of $f''(2)$.

13.8 L'Hôpital's Rule HL

This chapter began with the indeterminate form $\frac{0}{0}$, and resolved it by factorising. Factorising usually does not help when the functions are trigonometric, exponential or logarithmic.

There is a second method for those cases. If a limit gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$, the derivatives of the top and the bottom give the answer. The rule is named after the French mathematician Guillaume de l'Hôpital, and the name is said *lo-pee-TAL*, with a silent h.

KEY **L'Hôpital's rule.** If $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right of this equation exists.

Differentiate the top and the bottom *separately*. This is not the quotient rule. Then try the limit again, and if it is still indeterminate, apply the rule a second time.

The reason the rule works is short, and it explains why it needs the $\frac{0}{0}$ form.

Suppose $f(a) = 0$ and $g(a) = 0$. Then near $x = a$ each function is close to its own tangent line at a, so

$$f(x) \approx f'(a)(x-a), \qquad g(x) \approx g'(a)(x-a).$$

Divide one by the other. The factor $(x-a)$ appears in both, so it cancels:

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}.$$

So the quotient of the functions approaches the quotient of their gradients. This argument covers the $\frac{0}{0}$ case: only when both functions equal zero at a does each one reduce to a multiple of $(x-a)$, and only then does the cancelling work.

The limit $\lim_{x\to 0} \dfrac{\cos x}{x}$ shows what happens when they do not. Here $\cos 0 = 1$, not 0, so the form is $\frac{1}{0}$, which is not indeterminate and not one the rule covers. Differentiating top and bottom anyway would give

$$\lim_{x\to 0} \frac{-\sin x}{1} = 0,$$

and that is wrong: the limit does not exist, because the fraction increases without bound on one side of 0 and decreases without bound on the other. Check the form before you differentiate.

Worked through. In $\lim_{x\to 0} \frac{\sin x}{x}$, substituting gives $\frac{0}{0}$, so differentiate top and bottom:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1.$$

This confirms the limit stated in §13.1. In the same way, substituting into $\lim_{x\to 0} \frac{\ln(1+x)-x}{x^2}$ gives $\frac{0}{0}$. One application gives $\dfrac{1/(1+x)-1}{2x}$, still $\frac{0}{0}$, so apply the rule again:

$$\lim_{x\to 0} \frac{\ln(1+x)-x}{x^2} = \lim_{x\to 0} \frac{\dfrac{1}{1+x}-1}{2x} = \lim_{x\to 0} \frac{-\dfrac{1}{(1+x)^2}}{2} = -\frac{1}{2}.$$

CAUTION L'Hôpital's rule applies *only* to the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Always check that the limit is indeterminate before differentiating. Applying it to a form like $\frac{2}{0}$ or $\frac{3}{5}$ can give a wrong answer.

EXAMPLE 13.8

Evaluate $\lim_{x\to\infty} \dfrac{\ln x}{x}$.

As $x \to \infty$ this is $\frac{\infty}{\infty}$. Differentiate top and bottom:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

The denominator increases without bound while the numerator increases very slowly, so the ratio approaches zero. L'Hôpital's rule makes that comparison exact.

YOUR TURN 13.8 L'Hôpital's Rule HL

Applying the rule once

60. Evaluate each limit, checking first that it is indeterminate.

(a) $\lim_{x\to 0} \dfrac{e^x - 1}{x}$

(b) $\lim_{x\to 0} \dfrac{\sin 2x}{x}$

(c) $\lim_{x\to 0} \dfrac{\tan x}{x}$

(d) $\lim_{x\to\infty} \dfrac{\ln x}{x^2}$

(e) $\lim_{x\to 1} \dfrac{\ln x}{x - 1}$

61. Evaluate $\lim_{x\to\infty} \dfrac{3x^2 + x}{2x^2 - 5}$ in two ways: by dividing by the highest power, and by L'Hôpital's rule. State which you would use in an examination, and why.

Applying the rule twice

62. Each of these needs the rule applied twice. Evaluate them.

(a) $\lim_{x\to 0} \dfrac{1 - \cos 2x}{x^2}$

(b) $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$

(c) $\lim_{x\to\infty} \dfrac{x^2}{e^x}$

Checking the form first

63. A student evaluates $\lim_{x\to 0} \dfrac{x + 2}{x + 1}$ by differentiating top and bottom, obtaining 1. Explain the error and give the correct value.

64. State the two indeterminate forms to which L'Hôpital's rule applies, and explain why the rule must not be used on a form such as $\frac{2}{0}$.

Concept check

65. Explain why L'Hôpital's rule requires the derivatives of the top and the bottom taken **separately**, and not the derivative of the quotient.

Chapter 13 · Limits & Derivatives

Reflections

TOK The early calculus was built on "infinitesimals," quantities treated as nonzero when dividing and as zero when convenient. The philosopher George Berkeley mocked them in 1734 as "ghosts of departed quantities," and his criticism was fair: the logic was not secure. The method gave correct answers for about 150 years before the limit gave it a secure logical foundation, and the limit brought a puzzle of its own: $\frac{0}{0}$ has no value, yet a limit of that form often has one exactly. Is a mathematical technique justified by its results, or only by its proof? And where does the meaning of a limit sit: in the symbols, or in the process that produced them?

IM Calculus was created twice, independently. Isaac Newton developed his "method of fluxions" in England in the 1660s; Gottfried Leibniz developed his version, with the $\frac{dy}{dx}$ notation we still use, in Paris in the 1670s. A long dispute over who was first divided European mathematics for about a century. Newton's notation was used in Britain and Leibniz's in the rest of Europe. British mathematics progressed more slowly in that period, partly because it kept the less convenient symbols. Notation is not neutral: a good symbol can make a whole subject easier to work in.

RW The derivative measures change, so it appears in any subject that studies how one quantity changes with another. Velocity and acceleration in physics, marginal cost in economics, reaction rates in chemistry, growth rates in biology: each is a derivative. When a report says prices are 'rising but more slowly', it is making a claim about a second derivative: prices are increasing, but their rate of increase is falling.

EXT L'Hôpital's rule is related to a deeper idea. Near a point, many functions are well approximated by a polynomial, the Maclaurin series: $\sin x \approx x - \frac{x^3}{6}$, $e^x \approx 1 + x + \frac{x^2}{2}$. Substituting these into an indeterminate limit resolves it without differentiating repeatedly. The same problem that L'Hôpital's rule solves leads to representing functions as infinite polynomials, an important idea in analysis Ch. 16.

End-of-Chapter Problems
Chapter 13: Limits & Derivatives

1. Differentiate each of the following, simplifying your answers.

(a) $y = (2x^3 - 1)^4$ [2]

(b) $y = x^2 \ln x$ [2]

(c) $y = \dfrac{x^2}{\cos x}$ [2]

(d) $y = e^{3x} \cos x$ [3]

(e) $y = \sqrt{4x^2 + 1}$ [2]

(f) $y = \dfrac{e^x}{x^2 + 1}$ [3]

2. Differentiate each of the following, simplifying where possible.

(a) $y = (3x^2 + 1)^5$ [2]

(b) $y = x^3 e^{-x}$ [2]

(c) $y = \dfrac{x^2}{x - 2}$ [3]

(d) $y = \ln(3x^2 + 2)$ [2]

(e) $y = \dfrac{\cos x}{x^2}$ [3]

(f) $y = (x + 1)^3 (2x - 5)^4$ [3]

3. Differentiate each of the following.

(a) $y = \arctan(3x)$ [2]

(b) $y = \sec(2x)$ [2]

(c) $y = \log_3(x^2 + 1)$ [2]

(d) $y = \arcsin(x^2)$ [2]

(e) $y = \operatorname{cosec}(x^2)$ [2]

(f) $y = \cot(3x)$ [2]

4. Differentiate each of the following.

(a) $y = \sec^2 x$ [2]

(b) $y = \ln(\cos x)$ [3]

(c) $y = x \arctan x$ [3]

(d) $y = \arccos(2x)$ [2]

(e) $y = e^{\sin x}$ [2]

(f) $y = \dfrac{\arctan x}{x}$ [3]

5. Evaluate $\displaystyle\lim_{x \to 0} \frac{e^{2x} - 1}{x}$. [2]

6. Differentiate $y = e^x \sin x$, and hence find the gradient of the curve at $x = 0$. [3]

7. Find the equation of the tangent to $y = x^2$ that is parallel to the line $y = 6x - 1$. [4]

8. Using the quotient rule on $\tan x = \dfrac{\sin x}{\cos x}$, prove that $\dfrac{d}{dx} \tan x = \sec^2 x$. [4]

9. Consider $f(x) = x^3 - 3x$.

(a) Find $f'(x)$. [1]

(b) Find the coordinates of the points where the tangent is horizontal. [4]

10. Find the coordinates of the points on the curve $y = x^3 - 3x^2$ where the tangent is parallel to the line $y = 9x + 1$. [4]

11. Consider $f(x) = x^2 - 2x$.

(a) Find $f'(x)$ from first principles. [4]
(b) Find the equation of the tangent to the curve $y = f(x)$ at the point where $x = 3$. [3]
(c) Find the equation of the normal at the same point. [2]

12. Consider $f(x) = |x - 2|$.

(a) Sketch the graph of f. [1]
(b) Explain briefly why f is continuous at $x = 2$. [2]
(c) Explain why f is not differentiable at $x = 2$. [2]

13. GDC The function $f(x) = \dfrac{2^x - 1}{x}$ is not defined at $x = 0$.

(a) Copy and complete the table of values of $f(x)$, giving 5 significant figures:
$x = -0.1,\ -0.01,\ 0.01,\ 0.1.$ [2]
(b) Suggest the value of $\lim_{x\to 0} f(x)$, correct to 3 significant figures. [1]
(c) Use l'Hôpital's rule to find the exact value of the limit, and verify that it agrees with your answer to part (b). [2]

14. A curve has equation $y = \dfrac{\ln x}{x}$ for $x > 0$.

(a) Find $\dfrac{dy}{dx}$. [3]
(b) Show that the curve has a horizontal tangent at $x = e$. [2]

15. A curve has equation $y = \dfrac{e^{2x}}{1 + x^2}$.

(a) Find $\dfrac{dy}{dx}$. [4]
(b) Find the gradient of the curve at $x = 0$. [1]

16. Consider the curve $y = e^x$ at the point where $x = 1$.

(a) Show that the tangent there passes through the origin. [4]
(b) Find the equation of the normal at the same point. [2]

17. The displacement of a particle is $s(t) = t^3 - 6t^2 + 9t$ metres at time t seconds, for $t \geq 0$.

(a) Find the velocity $v(t) = s'(t)$. [2]
(b) Find the times at which the particle is momentarily at rest. [3]
(c) Find the acceleration at $t = 3$. [2]

18. GDC A cup of coffee is left to cool. Its temperature t minutes after it is poured is modelled by $T(t) = 20 + 70e^{-0.05t}$ degrees Celsius, for $t \geq 0$.

(a) Find the rate of change of T at $t = 10$, giving your answer to 3 significant figures, and state what its sign tells you. [2]
(b) Find the time at which the coffee is cooling at a rate of 1 degree Celsius per minute. [2]
(c) The tangent to the graph of T at $t = 10$ is used to estimate the temperature at $t = 15$. Find this estimate and the value given by the model, each to 3 significant figures, and explain why the estimate is lower. [3]

19. Evaluate $\lim_{x \to 0} \dfrac{e^{2x} - 1 - 2x}{x^2}$. [3]

20. Evaluate $\lim_{x \to 0} \dfrac{x - \sin x}{x^3}$, justifying each application of l'Hôpital's rule. [5]

21. Differentiate each function from first principles, showing every step of the limit.

(a) $f(x) = \sqrt{x}$, for $x > 0$. *Multiply the difference quotient by the conjugate.* [4]
(b) $f(x) = e^x$. You may use $\lim_{h \to 0} \dfrac{e^h - 1}{h} = 1$. [3]
(c) $f(x) = \sin x$. You may use the compound-angle formula for $\sin(x + h)$, together with $\lim_{h \to 0} \dfrac{\sin h}{h} = 1$ and $\lim_{h \to 0} \dfrac{\cos h - 1}{h} = 0$. [5]

22. GDC The profile of a hillside is modelled by $y = 4xe^{-0.5x}$ for $0 \leq x \leq 12$, where x and y are measured in metres.

(a) Find $\dfrac{dy}{dx}$. [2]
(b) Find the equation of the tangent to the profile at the point where $x = 1$, and show that it passes through the point $(-1, 0)$. [3]
(c) Find the two values of x at which the gradient of the hillside is -0.5, giving your answers to 3 significant figures. [3]

23. GDC Let $f(x) = x \cos x$ for $0 \leq x \leq \pi$.

(a) Find $f'(x)$. [2]
(b) Show that the tangent to the graph of f is horizontal where $x \tan x = 1$. [2]
(c) Find the coordinates of the point on the graph of f where the tangent is horizontal, giving your answers to 3 significant figures. [2]
(d) Find the equation of the normal to the graph of f at the point where $x = 2$. Give your answer in the form $y = mx + c$, with m and c correct to 3 significant figures. [3]

24. Let $f(x) = xe^{2x}$.

(a) Show that $f'(x) = (2x+1)e^{2x}$ and $f''(x) = 4(x+1)e^{2x}$. [3]
(b) Find $f'''(x)$. [2]
(c) Suggest an expression for $f^{(n)}(x)$, the nth derivative of f, for every positive integer n. [2]
(d) Prove your conjecture by mathematical induction. [5]
(e) Show that $y = f(x)$ satisfies $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$. [2]

25. Find the equations of *both* tangents to the curve $y = x^2$ that pass through the point $(0, -1)$. [6]

26. Let $L = \lim_{x\to 0} \dfrac{\tan x - x}{x^3}$.

(a) Show that one application of l'Hôpital's rule gives $L = \lim_{x\to 0} \dfrac{\tan^2 x}{3x^2}$. [2]
(b) Hence find the value of L. [3]

27. The function f is defined by $f(x) = \dfrac{1-\cos x}{x}$ for $x \neq 0$, and $f(0) = 0$.

(a) Use l'Hôpital's rule to show that f is continuous at $x = 0$. [2]
(b) Hence, using the definition of the derivative, show that $f'(0) = \dfrac{1}{2}$. [4]
(c) Find $f'(x)$ for $x \neq 0$. [3]
(d) Find the equation of the tangent to the graph of f at the point where $x = \pi$, giving exact coefficients. [3]
(e) Show that this tangent meets the x-axis at $x = 2\pi$. [1]

28. Let $g(x) = \dfrac{2x-1}{x+2}$, $x \neq -2$.

(a) Write down the equations of the vertical and horizontal asymptotes. [2]
(b) Use the quotient rule to show that $g'(x) = \dfrac{5}{(x+2)^2}$. [3]
(c) Hence explain why g is increasing on every interval of its domain. [2]
(d) Find the equation of the tangent to the curve at the point where $x = 0$. [2]
(e) Find the equation of the normal at the same point. [2]

Challenge Questions

29. **The power rule, proved.** Section 13.3 stated the power rule after two examples suggested it. Here you prove it for every positive integer n, using the binomial theorem, and then look at how far the argument reaches beyond the positive integers.

(a) Let $n \geq 2$ be an integer. Write down the first three terms of $(x+h)^n$ in ascending powers of h. [3]

(b) Hence show that, for every integer $n \geq 2$,

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + {}^nC_2 x^{n-2}h + \cdots + h^{n-1},$$

and explain why every term after the first still contains h. [5]

(c) Deduce that $\frac{d}{dx}x^n = nx^{n-1}$ for every integer $n \geq 2$. Check separately that the result also holds for $n = 1$. [3]

(d) Check your working by expanding in full for $n = 4$. [3]

(e) For $n = -1$ and $x \neq 0$, the binomial series of Chapter 2 gives $(x+h)^{-1} = x^{-1}\left(1 + \frac{h}{x}\right)^{-1}$ as an infinite series, valid for $|h| < |x|$. Write down its first three terms, and hence the first two terms of $\frac{(x+h)^{-1} - x^{-1}}{h}$. Explain why the step used in part (c) is no longer enough on its own. Then confirm that $\frac{d}{dx}x^{-1} = -x^{-2}$ by differentiating $\frac{1}{x}$ from first principles without a series. [4]

(f) The rule also holds for fractional powers. For $x > 0$, prove it for $n = \frac{1}{2}$ by multiplying the numerator and denominator of $\frac{\sqrt{x+h} - \sqrt{x}}{h}$ by $\sqrt{x+h} + \sqrt{x}$. Then investigate how the factorisation

$$a^q - b^q = (a-b)\left(a^{q-1} + a^{q-2}b + \cdots + b^{q-1}\right)$$

extends the method to $n = \frac{1}{q}$ for any positive integer q, and how you might then reach every positive rational power. [3]

30. Where the derivative of sine comes from. The derivatives of $\sin x$ and $\cos x$ in §13.5 rest on two limits. Here you prove the first limit geometrically, deduce the second, and use both to prove that the derivative of $\sin x$ is $\cos x$. Angles are in radians throughout. You may use the *squeeze principle*: if $g(x) \leq f(x) \leq k(x)$ for all x close to a (with $x \neq a$), and $g(x) \to L$ and $k(x) \to L$ as $x \to a$, then $f(x) \to L$ as well.

(a) Let $0 < x < \frac{\pi}{2}$. On the unit circle with centre O, let $A = (1, 0)$ and $P = (\cos x, \sin x)$, and let the line OP meet the tangent to the circle at A at the point T. Show that $T = (1, \tan x)$. By comparing the areas of triangle OAP, sector OAP and triangle OAT, show that

$$\sin x < x < \tan x.$$

[4]

(b) Hence show that $\cos x < \frac{\sin x}{x} < 1$ for $0 < x < \frac{\pi}{2}$, and explain why the same inequality holds for $-\frac{\pi}{2} < x < 0$. Deduce that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. [4]

(c) Show that $\frac{\cos x - 1}{x} = \frac{-\sin^2 x}{x(\cos x + 1)}$. Hence show that $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$. *(Hint: multiply* *above and below by* $\cos x + 1$) [4]

(d) The compound-angle identity gives $\sin(x+h) = \sin x \cos h + \cos x \sin h$. Use it to show that

$$\frac{\sin(x+h) - \sin x}{h} = \sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right).$$

[3]

(e) Hence prove from first principles that $\dfrac{d}{dx}\sin x = \cos x$. [3]

(f) Section 13.8 evaluates $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$ in one line by L'Hôpital's rule. Explain why that calculation cannot serve as a *proof* of the limit. [2]

(g) Put $x = \frac{\pi}{n}$ in part (a), where $n \geq 3$ is an integer, to show that $n \sin\frac{\pi}{n} < \pi < n \tan\frac{\pi}{n}$, and interpret the two bounds as half the perimeters of regular n-sided polygons drawn inside and outside a circle of radius 1. Archimedes used these bounds with $n = 96$. Investigate how quickly the bounds close in on π as n is doubled, and how the bounds for $2n$ sides can be found from those for n sides without knowing π. [3]

31. When L'Hôpital's rule says nothing. The rule in §13.8 carries a condition: $\lim \dfrac{f}{g} = \lim \dfrac{f'}{g'}$ *provided the limit on the right exists*. This investigation finds limits where that condition fails, and then uses the rule, correctly, to compare how fast functions grow.

(a) Consider $\lim\limits_{x\to\infty} \dfrac{x + \sin x}{x}$. Show that it is of the form $\frac{\infty}{\infty}$, and that applying the rule once gives a limit that does not exist. Find the limit by another method, and explain why this does not contradict the rule. [5]

(b) Consider $\lim\limits_{x\to\infty} \dfrac{\sqrt{1+x^2}}{x}$. Show that applying the rule once gives $\dfrac{x}{\sqrt{1+x^2}}$, and that applying it again returns the original fraction. Find the limit by another method. [4]

(c) Use the rule to evaluate $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x}$ for $n = 1$, 2 and 3. State a conjecture for every positive integer n, and prove it by induction on n. [5]

(d) Let $a > 0$. Show that $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^a} = 0$. [2]

(e) Parts (c) and (d) place $\ln x$, then every positive power x^a, then e^x, in increasing order of growth as $x \to \infty$. Investigate where the functions x^{100}, 1.01^x, $x \ln x$, $e^{\sqrt{x}}$ and x^x fit into this order, and give a precise meaning to the statement "f grows faster than g". [3]

32. How smooth is a join? A curve built from two pieces can meet itself without a jump and still have a corner. This investigation measures how smooth a join is by counting how many derivatives exist there. For each positive integer n define

$$f_n(x) = \begin{cases} x^n & x \geq 0, \\ 0 & x < 0. \end{cases}$$

(a) By finding the one-sided limits of $\dfrac{f_2(h) - f_2(0)}{h}$ as $h \to 0$, show that f_2 is differentiable at $x = 0$, and state $f_2'(0)$. [3]

(b) Write down $f_2'(x)$ for every real x. Show that f_2' is continuous at 0 but not differentiable there. [3]

(c) Investigate f_3 in the same way, and find how many times f_3 can be differentiated at $x = 0$. [3]

(d) Conjecture how many times f_n can be differentiated at $x = 0$, and prove your conjecture. [4]

(e) A straight railway track along the negative x-axis is joined at the origin to a track that follows $y = f_n(x)$ for $x \geq 0$. Where the track is nearly straight, y'' measures how sharply it bends, and a passenger feels that bending as a sideways push. Explain why the designer would choose $n = 3$ rather than $n = 2$. [2]

(f) Every f_n has only finitely many derivatives at 0. Investigate the function $g(x) = e^{-1/x}$ for $x > 0$, $g(x) = 0$ for $x \leq 0$. Use the substitution $t = \frac{1}{h}$ and Question 31(c) to show that $g'(0)$ exists, and then investigate whether g can be differentiated as many times as you like at $x = 0$. [3]