GEOMETRY & TRIGONOMETRY

CHAPTER 9

9

Vectors

SYLLABUS 3.12–3.18 HL ONLY

A camera tripod has three legs, not four. The choice is deliberate. Part of the reason is geometry.

Any three points lie in one flat surface, whatever their heights. All three feet of a tripod therefore reach the ground together, on any floor, however uneven. A fourth point need not lie in that surface. When it does not, the stand rocks between two pairs of legs. This is why a stool with four legs can wobble on an uneven floor, where a stool with three legs does not.

Ordinary arithmetic cannot express that fact. The fact is not about how large anything is. It is about position, direction and flatness. To state it, and to prove it, we need a language in which a direction is something we can calculate with.

A **vector** is a quantity that has both a size and a direction. Speed is a number, but velocity is a vector. Distance is a number, but displacement is a vector. A pilot who flies due north through a wind from the west does not travel due north, and does not travel at the aircraft's airspeed. The motion and the wind point in different directions, and both must be counted.

Once a quantity has a direction, addition changes. Three plus three need not give six. If the two vectors point the same way, the answer is six. If they point in opposite directions, the answer is zero. Any value between those two is possible, and the angle between them fixes which one.

By the end of this chapter you will add and scale vectors, use the scalar product to find angles and to test for perpendicularity, and use the vector product to find areas and normals. You will write the equation of a line and of a plane, decide how two lines or three planes are arranged, and find shortest distances. A vector holds a size and a direction in a single object, which makes it the natural tool for describing position and movement in three dimensions. Physics, computer graphics and navigation all rely on it, from the pilot's crosswind to the satellites overhead.

9.1 Vectors: Magnitude and Direction

A vector has exactly two properties: a **magnitude**, which is its length, and a **direction**. Where it sits in space does not matter. Two arrows of the same length pointing the same way are the same vector.

An arrow drawn to represent a vector is called a **directed line segment**. "Directed" because it has an arrowhead, and "line segment" because it is a piece of a line with two ends. Examination questions use this phrase; it means an arrow and nothing more.

Vectors must be kept apart from ordinary numbers, which in this context are called **scalars**. Three notations are in common use, and they all mean the same thing. Printed books, including this one, use a bold letter $\mathbf{a}$. By hand you cannot write bold type, so you write either a letter with an arrow above it, $\vec{a}$, or an underlined letter, $\underline{a}$. An arrow is also used for a vector named by its two endpoints: $\overrightarrow{AB}$ means the vector from the point A to the point B.

The usual representation is the **column vector**. It lists how far to travel along each axis to get from the tail to the head.

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}.$$

Here $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the **base vectors**: arrows of length 1 along the x-, y- and z-axes. The hat is there to record that each one has length 1, a point returned to in the next subsection. Many books print them without hats, as $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$; the two notations are equivalent, and the formula booklet uses only the column form. The column form and the base-vector form describe the same vector, so use whichever is convenient.

9.1.1 The algebra of arrows

Vectors are added **tip to tail**. Draw the second vector starting where the first one ends. The sum is the arrow from the start of the first to the end of the second. In components it is simple: everything happens coordinate by coordinate.

KEY

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}, \qquad k\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}.$$

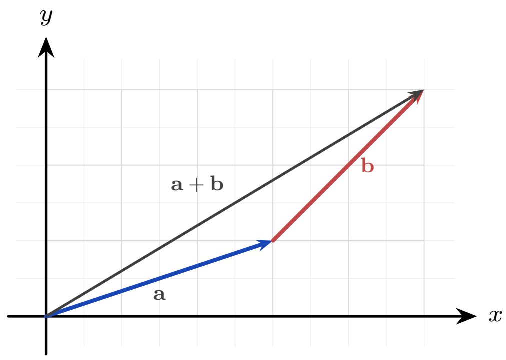


Figure 9.1 Adding tip to tail. Draw $\mathbf{b}$ starting where $\mathbf{a}$ ends; the sum runs from the start of $\mathbf{a}$ to the end of $\mathbf{b}$.

Scalar multiplication by k scales the length by $|k|$ and reverses the direction when k is negative. This gives a clean test: two nonzero vectors are **parallel** exactly when one is a scalar multiple of the other, $\mathbf{a} = k\mathbf{b}$.

Two particular vectors have names of their own.

The **negative** of $\mathbf{v}$, written $-\mathbf{v}$, is the vector of the same length pointing the opposite way. It is the case $k = -1$ of scalar multiplication. Subtraction is then just addition of the negative:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

The **zero vector** $\mathbf{0}$ has magnitude 0 and no direction at all. It is what you get from $\mathbf{v} + (-\mathbf{v})$, or from $0\mathbf{v}$. It is the only vector whose direction is undefined, and it is written in bold to keep it apart from the number 0.

Worked through. Take $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$. To find $2\mathbf{a} - \mathbf{b}$, scale first, then subtract, working one coordinate at a time:

$$2\mathbf{a} - \mathbf{b} = \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ -5 \end{pmatrix}.$$

Is it parallel to $\mathbf{c} = \begin{pmatrix} -8 \\ -12 \\ 10 \end{pmatrix}$, or to $\mathbf{d} = \begin{pmatrix} 12 \\ 18 \\ -14 \end{pmatrix}$? Parallel means one vector is a scalar multiple of the other, so look for a single k that works in all three coordinates. For $\mathbf{c}$, the first coordinate forces $k = -2$, and that value holds throughout:

$$-2\begin{pmatrix} 4 \\ 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -8 \\ -12 \\ 10 \end{pmatrix} = \mathbf{c},$$

so $\mathbf{c}$ is parallel to $2\mathbf{a} - \mathbf{b}$, pointing the opposite way because k is negative. For $\mathbf{d}$, the first two coordinates both give $k = 3$, but $3 \times (-5) = -15$, and the third coordinate of $\mathbf{d}$ is -14. No single k works, so $\mathbf{d}$ is not parallel to $2\mathbf{a} - \mathbf{b}$. Test all three coordinates before answering. If two coordinates give the same k and the third does not, the vectors are not parallel.

9.1.2 Magnitude and unit vectors

Three words are used for the same quantity. The **magnitude** of a vector, its **length**, and its **modulus** all mean the size of the arrow, and all are written $|\mathbf{v}|$. These notes use whichever reads more naturally in the sentence. The value comes straight from Pythagoras, extended into three dimensions.

KEY · IB The magnitude of $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

Dividing a vector by its own length leaves the direction untouched but sets the length to 1. The result is a **unit vector**, written $\hat{\mathbf{v}}$, the pure direction stripped of any magnitude, ready to be scaled to any length you like.

KEY $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ is the unit vector in the direction of $\mathbf{v}$.

The base vectors are unit vectors, which is what their hats record: $|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = |\hat{\mathbf{k}}| = 1$, one unit along each axis. So writing $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ says that $\mathbf{a}$ is built by taking a_1 steps of length 1 along x, then a_2 along y, then a_3 along z.

Worked through. To find a vector of magnitude 14 in the direction of $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$, first find the length: $|\mathbf{v}| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7$. The unit vector is $\hat{\mathbf{v}} = \frac{1}{7}\mathbf{v}$, and scaling it to length 14 means multiplying by 14:

$$14\hat{\mathbf{v}} = \frac{14}{7}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 12 \end{pmatrix}.$$

Reduce to a unit vector first, then rescale. Those two steps answer every question of the form "find a vector of magnitude m in the direction of".

YOUR TURN 9.1 Vectors: Magnitude and Direction

Magnitude and combining vectors

1. Find the magnitude of each vector.

(a) $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

2. Given $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$, evaluate each of the following.

(a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{a} - \mathbf{b}$

(c) $\mathbf{a} + 2\mathbf{b}$ (d) $3\mathbf{a} - \mathbf{b}$

Base vectors

3. Work with the base vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.

(a) Write $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ as a column vector.

(b) Write $\begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$ in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.

(c) Given $\mathbf{a} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + 3\hat{\mathbf{k}}$, find $2\mathbf{a} - \mathbf{b}$ in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, and find its magnitude.

Unit vectors and scaling

4. Unit vectors and scaling.

(a) Find the unit vector in the direction of $\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$.

(b) Find the unit vector in the direction of $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

(c) Find a vector of magnitude 6 in the direction of $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

(d) Explain why $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ has magnitude 1 for every non-zero vector $\mathbf{v}$.

Parallel vectors

5. Decide whether each pair of vectors is parallel, and justify your answer.

(a) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$

(c) $\begin{pmatrix} 3 \\ -6 \\ 9 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$

(d) Find p and q so that $\begin{pmatrix} 2 \\ p \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 6 \\ q \end{pmatrix}$ are parallel.

Concept check

6. A student says: "$|\mathbf{a} + \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$."

(a) Give a counter-example.

(b) State when the claim is true.

9.2 Position Vectors and Geometry

Fix the origin O. The **position vector** of a point A is the arrow $\overrightarrow{OA}$ from the origin to A; its components are simply the coordinates of A. Position vectors connect the algebra of vectors to the fixed points of a geometry problem.

The arrow from one point to another is found by subtraction. Much of the chapter uses this fact.

> KEY $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, the position vector of the head minus that of the tail.

The distance between A and B is then the magnitude of that displacement, $|\overrightarrow{AB}| = |\mathbf{b} - \mathbf{a}|$, and the midpoint of AB has position vector $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.

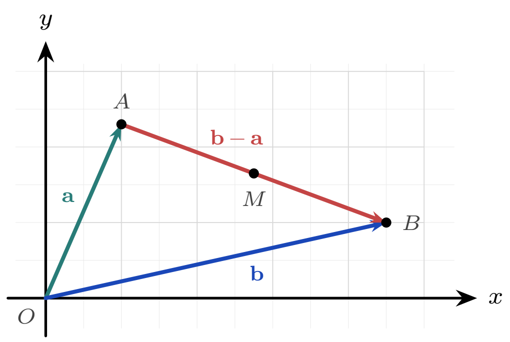


Figure 9.2 Head minus tail. The two position vectors run from the origin; the arrow between the points closes the triangle, so $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$. The midpoint M sits at the average of the two position vectors, $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.

Worked through. For the points $A(2, -3, 1)$ and $B(5, 1, 6)$,

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \qquad AB = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2}.$$

Head minus tail gives the arrow; the magnitude of that arrow is the distance. Every distance between two points reduces to those two steps.

Three points A, B, C are **collinear** (on one straight line) when $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel, that is, one is a scalar multiple of the other. This is the vector version of "same direction from a shared point", and the equation of a line in the next section is built from it.

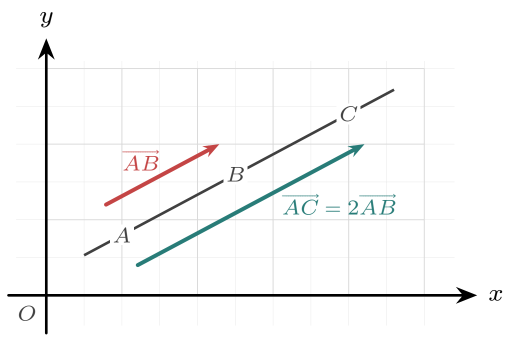


Figure 9.3 Collinear points. Both displacements start at A, and one is a scalar multiple of the other, so the three points lie on a single line. The two arrows are drawn slightly to either side of that line so they can be told apart.

9.2.1 Proving geometry with vectors

Vectors give a way of proving geometric facts by algebra, with no diagram-chasing and no coordinates. The method has three steps.

KEY **Proving a geometric property with vectors.**

1. Choose one point as the origin, and name the position vectors of the others with single letters.
2. Write every point in the figure in terms of those letters.
3. Show the required relation by algebra.

Choosing the origin well is what keeps the algebra short. Put it at a vertex that many edges meet.

Worked through. To prove that the diagonals of a parallelogram bisect each other, let the parallelogram be $OABC$, with the origin at O. Write $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

Because $OABC$ is a parallelogram, $\overrightarrow{CB} = \overrightarrow{OA} = \mathbf{a}$, so the fourth vertex is

$$\overrightarrow{OB} = \mathbf{a} + \mathbf{c}.$$

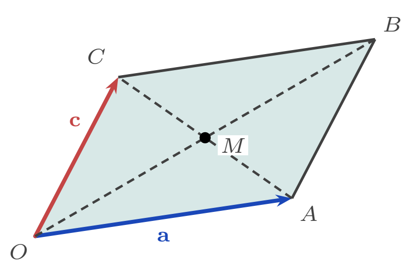


Figure 9.4 The parallelogram $OABC$. The two dashed diagonals meet at the point M, and the proof below shows that M is the midpoint of both.

The two diagonals are OB and AC. Find the midpoint of each.

Midpoint of OB: $\frac{1}{2}(\mathbf{0}+\mathbf{a}+\mathbf{c}) = \frac{1}{2}(\mathbf{a}+\mathbf{c})$.

Midpoint of AC: $\frac{1}{2}(\mathbf{a}+\mathbf{c})$.

The two midpoints are the same point. A single point lying halfway along both diagonals is exactly what "bisect each other" means. ■

Notice that no coordinates were used, and the result holds for every parallelogram, in a plane or in space.

EXAMPLE 9.1

M and N are the midpoints of the sides OA and OB of a triangle. Prove that MN is parallel to AB and half its length.

Take the origin at O, with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The midpoints are then

$$\overrightarrow{OM} = \tfrac{1}{2}\mathbf{a}, \qquad \overrightarrow{ON} = \tfrac{1}{2}\mathbf{b}.$$

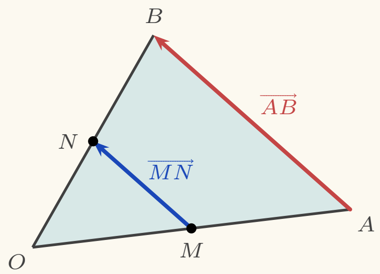


Figure 9.5 The midpoint theorem. Joining the midpoints of two sides gives a segment parallel to the third side and half as long.

Now find the two displacement vectors, each as head minus tail:

$$\overrightarrow{MN} = \tfrac{1}{2}\mathbf{b} - \tfrac{1}{2}\mathbf{a} = \tfrac{1}{2}(\mathbf{b}-\mathbf{a}), \qquad \overrightarrow{AB} = \mathbf{b}-\mathbf{a}.$$

So $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AB}$.

One vector is a scalar multiple of the other, so the two segments are parallel, and the scalar is $\frac{1}{2}$, so MN is half the length of AB. ■

That is the entire proof. This result is known as the midpoint theorem.

YOUR TURN 9.2 Position Vectors and Geometry

Vectors between points

7. Points $A(1, 2, -1)$ and $B(4, 0, 3)$ are given.

(a) Find $\overrightarrow{AB}$.
(b) Find the distance AB.
(c) Find the midpoint of AB.
(d) Find $\overrightarrow{BA}$.

8. Further position-vector work.

(a) A is $(1, 1, 1)$ and $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$. Find the coordinates of B.
(b) B is the midpoint of AC, with $A(2, 1, 3)$ and $B(4, 5, 3)$. Find C.
(c) Show that $A(1, 1, 1)$, $B(2, 3, 5)$ and $C(4, 7, 13)$ are collinear.

Proving geometry with vectors

9. $OABC$ is any quadrilateral. Take the origin at O and write $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$. The points P, Q, R and S are the midpoints of $[OA]$, $[AB]$, $[BC]$ and $[CO]$.

(a) Write the position vectors of P, Q, R and S in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.
(b) Find $\overrightarrow{PQ}$ and $\overrightarrow{SR}$.
(c) Hence prove that $PQRS$ is a parallelogram.

10. In triangle OAB, take the origin at O and write $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of $[AB]$, N is the midpoint of $[OA]$, and G is the point on $[OM]$ with $OG = \frac{2}{3}OM$.

(a) Show that $\overrightarrow{OG} = \frac{1}{3}(\mathbf{a} + \mathbf{b})$.
(b) Find $\overrightarrow{BG}$ and $\overrightarrow{BN}$, and hence prove that B, G and N are collinear.

Concept check

11. For $A(1, 2, -1)$ and $B(4, 0, 3)$, a student writes $\overrightarrow{AB} = \mathbf{a} - \mathbf{b} = \begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix}$. Identify the error and find $\overrightarrow{AB}$.

9.3 The Scalar Product

There are two ways to multiply vectors, and they answer two different questions. The first, the **scalar product** (or dot product), takes two vectors and returns a number, and that number carries the angle between them. It has two forms, and they always agree.

KEY · IB **Scalar product.** For $\mathbf{a}, \mathbf{b}$ with angle θ between them,

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad \text{and} \qquad \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta.$$

In the second form, the $\cos\theta$ is not decoration. Ordinary multiplication assumes both quantities are measured along the same line: 3×4 makes sense because both numbers sit on one number line. Two vectors usually point different ways, so multiplying their lengths would measure nothing in particular.

The repair is to bring them onto the same line first. Keep $\mathbf{a}$ as it is, and use only the part of $\mathbf{b}$ that lies along $\mathbf{a}$: the **projection** of $\mathbf{b}$ onto $\mathbf{a}$, found by dropping a perpendicular from the tip of $\mathbf{b}$, which has length $|\mathbf{b}| \cos\theta$. Multiplying those two gives

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \cos\theta,$$

the length of $\mathbf{a}$ times how much of $\mathbf{b}$ goes in $\mathbf{a}$'s direction. Doing it the other way round, projecting $\mathbf{a}$ onto $\mathbf{b}$, gives $|\mathbf{b}| \times |\mathbf{a}| \cos\theta$, the same number. That is why the order does not matter.

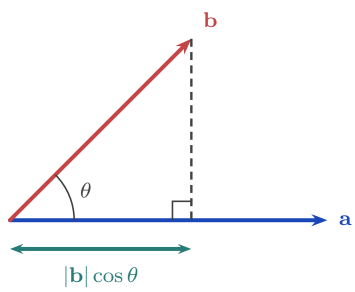


Figure 9.6 The projection of $\mathbf{b}$ onto $\mathbf{a}$. Dropping the tip of $\mathbf{b}$ perpendicularly onto $\mathbf{a}$ leaves a length of $|\mathbf{b}| \cos\theta$, and the scalar product multiplies that by $|\mathbf{a}|$.

The scalar product then obeys the rules you would expect of a multiplication.

- **Commutative:** $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$. The order does not matter.
- **Distributive:** $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$. Brackets expand as usual.
- **Scalars move outside:** $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b})$. A constant factor can be taken out.
- **Dotted with itself:** $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$, since $\theta = 0$ and $\cos 0 = 1$. The result is the squared length, and several proofs in this chapter use it.

The base vectors show both extremes at once. Each has length 1 and lies along its own axis, so dotting one with itself gives $1 \times 1 \times \cos 0 = 1$. Different axes are at right angles, so dotting two different base vectors gives 0.

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1, \qquad \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0.$$

This is exactly why the component form works. Expanding $(a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}) \cdot (b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}})$ produces nine terms, and every term pairing two *different* base vectors is zero, because of that zero dot product. Only the three matching terms survive, leaving $a_1b_1 + a_2b_2 + a_3b_3$.

The component form is easy to compute; the cosine form carries the geometry. Setting the two equal is what lets you find the angle between any two directions, and it is used throughout the chapter.

KEY · IB $\cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|\mathbf{a}|\,|\mathbf{b}|}.$

Worked through. To find the angle between $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$, compute the three ingredients: $\mathbf{a} \cdot \mathbf{b} = 6 + 12 + 18 = 36$, $|\mathbf{a}| = \sqrt{9 + 4 + 36} = 7$, $|\mathbf{b}| = 7$. Then

$$\cos\theta = \frac{36}{49} \implies \theta = \arccos\frac{36}{49} \approx 42.7^\circ.$$

Dot product over product of lengths: the recipe never changes, whether the vectors come from a triangle, a pair of lines, or a pair of planes.

A special case used often is a right angle. Since $\cos 90^\circ = 0$, the scalar product of perpendicular vectors is zero, and the converse holds too. This gives a test for perpendicularity that needs no angles at all, just a multiply-and-add.

KEY $\mathbf{a} \perp \mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} = 0 \quad$ (for nonzero $\mathbf{a}, \mathbf{b}$).

Parallel vectors sit at the opposite extreme. There θ is 0 or π, so $\cos\theta$ is $+1$ or -1 and the scalar product reaches the largest size it can. Taking the modulus covers both directions at once.

KEY For nonzero $\mathbf{a}, \mathbf{b}$: $\quad \mathbf{a} \parallel \mathbf{b} \iff |\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|.$

So the scalar product recognises both extremes. It is zero when the vectors are perpendicular, and its size is as large as possible when they are parallel. Every angle in between falls somewhere on that scale.

EXAMPLE 9.2

Find the angle between $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

The dot product is $\mathbf{a} \cdot \mathbf{b} = (1)(2) + (2)(0) + (2)(-1) = 0$. Because it vanishes, the vectors are perpendicular: $\theta = 90^\circ$. Neither magnitude was needed: a zero dot product settles the angle at once.

YOUR TURN 9.3 The Scalar Product

Calculating the scalar product

12. Evaluate each scalar product.

(a) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

(d) $\begin{pmatrix} 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

13. Use the definition $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$.

(a) Given $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$ and an angle of 60° between them, find $\mathbf{a} \cdot \mathbf{b}$.

(b) Given $|\mathbf{a}| = 2$, $|\mathbf{b}| = 5$ and an angle of 120° between them, find $\mathbf{a} \cdot \mathbf{b}$.

(c) Explain why $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ for every vector $\mathbf{a}$.

(d) Given $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$ and $\mathbf{a} \cdot \mathbf{b} = -15$, state what this tells you about the directions of $\mathbf{a}$ and $\mathbf{b}$, and justify your answer.

Angles between vectors

14. GDC Find the angle between each pair of vectors.

(a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(b) $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(d) $\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$

Perpendicular vectors

15. Use the scalar product to find k.

(a) Find k so that $\begin{pmatrix} 2 \\ k \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 0$.

(b) Find the possible values of k for which $\begin{pmatrix} k \\ 1 \\ 2 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} k \\ -5 \\ 2 \end{pmatrix}$.

Proofs with the scalar product

16. A rhombus has sides $\mathbf{a}$ and $\mathbf{b}$ drawn from one vertex, so $|\mathbf{a}| = |\mathbf{b}|$.

(a) Write the two diagonals in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) Expand $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$.

(c) Hence prove that the diagonals of a rhombus are perpendicular.

(d) Verify your result for $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

Concept check

17. For $A(3,1,0)$, $B(1,1,0)$ and $C(2,2,0)$, a student finds angle $A\hat{B}C$ using $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and obtains 135°. Identify the error and find the correct angle.

9.4 The Vector Equation of a Line

A line is fixed by two pieces of information: one point it passes through, and a direction it runs along. Start at a known point $\mathbf{a}$ and move any distance along a direction $\mathbf{b}$. As the **parameter** λ takes every real value, this reaches every point on the line.

> KEY · IB **Vector equation of a line:** $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where $\mathbf{a}$ is the position vector of a point on the line and $\mathbf{b}$ is a direction vector.

This differs from the equation of a line met earlier, and the difference matters. An equation such as $y = 2x + 1$ is a condition: a point either satisfies it or does not. The vector equation produces points: each value of λ gives one point of the line. That is why the vector form works in three dimensions, where a single equation in x, y and z describes a plane, not a line.

Writing $\mathbf{r} = (x, y, z)$ and reading off each coordinate gives the **parametric form**; eliminating λ between them gives the **Cartesian form**.

> KEY · IB **Parametric:** $x = x_0 + \lambda l, \quad y = y_0 + \lambda m, \quad z = z_0 + \lambda n.$
>
> **Cartesian:** $\dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}, \quad$ where $\mathbf{b} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$.

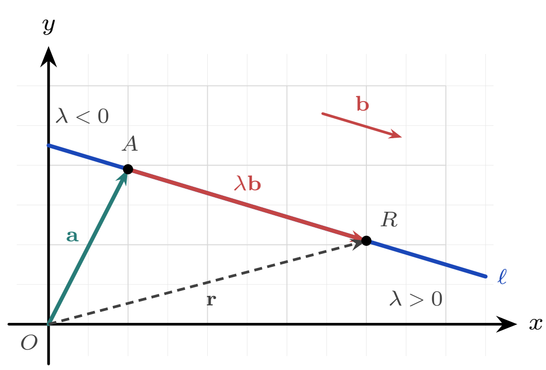


Figure 9.7 A line built from a point and a direction. The vector $\mathbf{a}$ reaches the known point A; travelling $\lambda\mathbf{b}$ from there lands on a general point R, whose position vector $\mathbf{r}$ closes the triangle. Negative λ runs back the other way, so the whole line is covered. A vector can be drawn anywhere, so the copy of $\mathbf{b}$ drawn above is the same vector.

The direction vector carries the geometry of how two lines relate. The **angle between two lines** is the angle between their direction vectors, found by the scalar product (take the acute value). Two lines are **parallel** when their directions are scalar multiples. And in three dimensions there is a possibility with no two-dimensional analogue: lines that are neither parallel nor crossing.

Two lines in space fall into exactly one of four cases. Two questions settle which: are the directions parallel, and do the lines share a point?

- **Coincident.** Directions parallel, every point shared. The same line written two ways.
- **Parallel.** Directions parallel, no point shared. Different lines running the same way.
- **Intersecting.** Directions not parallel, exactly one point shared. They cross once.
- **Skew.** Directions not parallel, no point shared. They pass without meeting, like an overpass above a road.

Only the last case is new. Two lines in a plane must either be parallel or cross, but in three dimensions they can miss each other entirely.

KEY **Classifying two lines.**

1. Compare the directions. If they are scalar multiples, check whether a point of one lies on the other: yes means **coincident**, no means **parallel**.
2. If the directions differ, set the two equations equal and solve for the parameters. Two components fix them; the third decides.
3. A consistent solution means the lines **intersect**, and substituting back gives the point of intersection. An inconsistent system means they are **skew**.

Worked through. Do the lines $\mathbf{r}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ intersect? The directions are not multiples of each other, so the lines are not parallel. Setting components equal:

$$2 + \lambda = 1 + 2\mu, \qquad 1 + \lambda = 3 - \mu, \qquad \lambda = 2 + \mu.$$

From the first two, $\lambda - 2\mu = -1$ and $\lambda + \mu = 2$, so $\mu = 1$, $\lambda = 1$. Check in the third: $\lambda = 1$ and $2 + 1 = 3$. Since $1 \neq 3$, the system is inconsistent, so the lines are **skew**. Two equations fix the parameters; the third decides whether the lines really meet.

9.4.1 Lines in motion: kinematics

So far the parameter has had no meaning attached to it. Give it one and the line equation describes a moving object. Read λ as *time* t, and $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ says: start at $\mathbf{a}$, and move with constant **velocity** $\mathbf{b}$. The direction of $\mathbf{b}$ is the heading; its magnitude $|\mathbf{b}|$ is the **speed**. This is the pilot from the opening page: the velocity is the tip-to-tail sum of the aircraft's velocity through the air and the wind velocity.

EXAMPLE 9.3

An aircraft passes the point $(20, 10)$ (coordinates in km) at $t = 0$ and flies with constant velocity $\begin{pmatrix} 240 \\ 100 \end{pmatrix}$ km h^{-1}. Find its speed, its position at $t = 2$ hours, and whether its path passes

through the airport at $(500, 210)$.
The speed is the magnitude of the velocity: $\sqrt{240^2+100^2} = 260$ km h^{-1}. The position at time t is

$$\mathbf{r} = \begin{pmatrix} 20 \\ 10 \end{pmatrix} + t\begin{pmatrix} 240 \\ 100 \end{pmatrix}, \qquad \text{so at } t = 2: \quad \mathbf{r} = \begin{pmatrix} 500 \\ 210 \end{pmatrix}.$$

That *is* the airport, reached at exactly $t = 2$. To test any point, solve the first coordinate for t, then check the others. One value of t must satisfy every equation at once. This is the same test used for intersecting lines.

For two objects moving at different constant velocities, the distance between them usually falls, reaches a least value, and then rises again. The moment of **closest approach** is found from the vector between them. Subtracting the position vectors gives $\mathbf{r}_B - \mathbf{r}_A$, whose components are linear in t, so the squared distance

$$d^2 = |\mathbf{r}_B - \mathbf{r}_A|^2$$

is a quadratic in t. Completing the square gives its least value and the time at which that value occurs.

Work with d^2 rather than with d. Squaring clears the square root, and the two are least at the same instant, since squaring preserves order among non-negative numbers.

Worked through. Two birds fly in straight lines at constant velocity. Taking t in seconds and distances in metres, their position vectors are

$$\mathbf{r}_A = \begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

To find the time at which they are closest, and the least distance between them, subtract, component by component:

$$\mathbf{r}_B - \mathbf{r}_A = \begin{pmatrix} 2-8 \\ 3-5 \\ 3-1 \end{pmatrix} + t\begin{pmatrix} 3-1 \\ 1-(-1) \\ 2-3 \end{pmatrix} = \begin{pmatrix} 2t-6 \\ 2t-2 \\ 2-t \end{pmatrix}.$$

Its squared length is

$$d^2 = (2t-6)^2+(2t-2)^2+(2-t)^2 = (4t^2-24t+36)+(4t^2-8t+4)+(t^2-4t+4) = 9t^2-36t+44.$$

Complete the square:

$$d^2 = 9\left(t^2-4t\right)+44 = 9(t-2)^2-36+44 = 9(t-2)^2+8.$$

The squared bracket is never negative and is zero only at $t = 2$, so the least value of d^2 is 8, reached at $t = 2$ seconds. The least distance is

$$d = \sqrt{8} = 2\sqrt{2} \approx 2.83 \text{ m}.$$

Check either side: at $t = 1$ and at $t = 3$ the formula gives $d^2 = 17$ both times, larger than 8 and equal to each other, which is what a parabola symmetric about $t = 2$ requires.

YOUR TURN 9.4 The Vector Equation of a Line

Writing the equation of a line

18. Writing the equation of a line.

(a) Write a vector equation of the line through $A(1, 2, 3)$ with direction $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

(b) Find a direction vector of the line through $A(1, 0, 2)$ and $B(4, 3, 5)$.

(c) Write a vector equation of that line.

19. Forms of the equation.

(a) Write $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ in parametric form.

(b) Write $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ in Cartesian form.

(c) Does $(3, 3, 2)$ lie on $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$? Justify your answer.

Classifying two lines

20. Decide whether each pair of lines is coincident, parallel, intersecting or skew. Where the lines intersect, find the point of intersection.

(a) $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix}$

(b) $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$

(c) $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ 5 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(d) $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Angle between two lines

21. GDC Find the acute angle between each pair of lines.

(a) $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ 5 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(b) $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and the x-axis, $\mathbf{r} = \mu \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Lines in motion

22. A boat leaves the point $(2, -1)$, in km, at $t = 0$ and moves with constant velocity $\binom{6}{8}$ km h^{-1}.

(a) Find the speed of the boat.
(b) Write down its position vector at time t.
(c) Find its position after 30 minutes.
(d) A lighthouse stands at $(20, 15)$. Show that the boat does not pass through it.

23. Two drones fly in straight lines at constant velocity. With t in seconds and distances in metres, their position vectors are

$$\mathbf{r}_A = \begin{pmatrix}1\\2\\0\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix}7\\0\\5\end{pmatrix} + t\begin{pmatrix}-1\\2\\1\end{pmatrix}.$$

(a) Show that $\mathbf{r}_B - \mathbf{r}_A = \begin{pmatrix}6-3t\\t-2\\5\end{pmatrix}$.
(b) Show that the square of the distance between the drones is $d^2 = 10t^2 - 40t + 65$.
(c) Find the time at which the drones are closest, and their least distance apart.

Concept check

24. A student says: "*the direction vectors of the two lines are not parallel, so the lines intersect.*" State when this is false, and describe the check that decides.

9.5 The Vector Product

The second way to multiply vectors returns not a number but a *vector*. The **vector product** (or cross product) $\mathbf{a} \times \mathbf{b}$ points perpendicular to *both* of the vectors it is built from, and its length measures the area they span. Where the dot product was about alignment, the cross product is about the plane the two vectors sweep out.

KEY · IB **Vector product.** For $\mathbf{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$,

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix}a_2b_3 - a_3b_2\\a_3b_1 - a_1b_3\\a_1b_2 - a_2b_1\end{pmatrix}, \qquad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

The magnitude and the direction can be combined into one statement,

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta \, \hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Taking magnitudes gives the second formula in the box, since $|\hat{\mathbf{n}}| = 1$.

The direction is fixed by the **right-hand screw rule**: turn a screw from $\mathbf{a}$ towards $\mathbf{b}$ and it advances along $\mathbf{a} \times \mathbf{b}$. Four properties are used constantly.

- **Perpendicular to both.** The result is at right angles to the plane of $\mathbf{a}$ and $\mathbf{b}$. This is what makes it the tool for building normal vectors in the next section.
- **Not commutative.** It is **anti-commutative**: $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$. The order matters, and reversing it reverses the answer.
- **Distributive, and scalars come out.** $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ and $(k\mathbf{v}) \times \mathbf{w} = k(\mathbf{v} \times \mathbf{w})$.
- **Zero for parallel vectors.** If $\mathbf{a}$ and $\mathbf{b}$ are parallel then $\sin\theta = 0$, so $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. In particular $\mathbf{v} \times \mathbf{v} = \mathbf{0}$, the counterpart of $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$.

The base vectors again show the pattern. Each is perpendicular to the other two, so crossing two different ones gives a unit vector along the third axis, and the screw rule fixes which way it points. Crossing one with itself gives $\mathbf{0}$, since the angle is 0 and $\sin 0 = 0$.

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \qquad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \qquad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}, \qquad \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}.$$

Read the first three in the cyclic order $\hat{\mathbf{i}} \to \hat{\mathbf{j}} \to \hat{\mathbf{k}} \to \hat{\mathbf{i}}$: going forwards gives a plus, and going backwards gives a minus, so $\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$.

9.5.1 One array, two products

Both products come out of the same nine numbers. Multiply every component of $\mathbf{a}$ by every component of $\mathbf{b}$ and lay the results in a square.

	b_1	b_2	b_3
a_1	a_1b_1	a_1b_2	a_1b_3
a_2	a_2b_1	a_2b_2	a_2b_3
a_3	a_3b_1	a_3b_2	a_3b_3

The **diagonal** is the scalar product. Those are the three terms where a base vector meets itself, and $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = 1$ keeps them; everything off the diagonal is removed by $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0$. Adding the shaded entries gives $a_1b_1 + a_2b_2 + a_3b_3$.

The **off-diagonal** entries are the vector product, taken in opposite pairs. Here the diagonal vanishes, since $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \mathbf{0}$, and each pair of mirror-image entries subtracts to give one component:

$$\underbrace{a_2b_3 - a_3b_2}_{\text{the pair missing } 1}, \qquad \underbrace{a_3b_1 - a_1b_3}_{\text{the pair missing } 2}, \qquad \underbrace{a_1b_2 - a_2b_1}_{\text{the pair missing } 3}.$$

So the component of $\mathbf{a} \times \mathbf{b}$ in a given direction is built from the two rows and columns that leave that direction out. That the result is perpendicular to both vectors is checked by the scalar product: $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ expands to six terms that cancel in pairs. The two products are the same array read two ways: down the diagonal, or across it.

You need not write the array out. For each component, cover its own row in both vectors and multiply the remaining four numbers crosswise. The subtraction then runs one way for the outer two components and the other way for the middle one:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

Only the middle component reverses. Covering row 1 leaves rows 2 and 3, and covering row 3 leaves rows 1 and 2; both pairs are in the cyclic order $1 \to 2 \to 3 \to 1$. Covering row 2 leaves rows 1 and 3, which run against that order, so the two products swap places before subtracting. Taking all three the same way is a common error here, and the vector it produces is usually not perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, so a dot-product check will usually catch it. A cross product built correctly answers "find a vector perpendicular to both" immediately; a *unit* vector perpendicular to both needs one further step.

Worked through. For $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, each component of $\mathbf{a} \times \mathbf{b}$ leaves out its own row, and the middle one subtracts the other way round:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} (1)(1) - (0)(2) \\ (0)(2) - (2)(1) \\ (2)(2) - (1)(2) \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}.$$

The middle entry is $a_3b_1 - a_1b_3$, not $a_1b_3 - a_3b_1$; reading it in the same direction as the other two would give $+2$ there instead of -2. Check it against both vectors: $(1)(2) + (-2)(1) + (2)(0) = 0$ and $(1)(2) + (-2)(2) + (2)(1) = 0$. The rejected version fails this test, since $(1)(2) + (2)(1) + (2)(0) = 4$. Perpendicular is not the same as unit. For a unit vector perpendicular to both, note that the magnitude is $\sqrt{1 + 4 + 4} = 3$, so divide by it:

$$\hat{\mathbf{n}} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}.$$

The vector $-\hat{\mathbf{n}} = \frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$ is equally correct. It is $\mathbf{b} \times \mathbf{a}$ divided by its magnitude, and it runs the opposite way along the same perpendicular line, so unless the question fixes a direction either answer is complete.

9.5.2 Area of a parallelogram and a triangle

The magnitude has a clean geometric meaning: it is the area of the parallelogram with $\mathbf{a}$ and $\mathbf{b}$ as sides. To see why, read the two factors of $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$ separately. Take $|\mathbf{a}|$ as the base. Then $|\mathbf{b}| \sin\theta$ is the perpendicular height, because $\mathbf{b}$ is the slanted side and $\sin\theta$ keeps only the part of it at right angles to the base. So the magnitude is base $\times$ height, which is the area.

> KEY · IB Area of a parallelogram $= |\mathbf{a} \times \mathbf{b}|$, where $\mathbf{a}$ and $\mathbf{b}$ are two adjacent sides.

A triangle is half of such a parallelogram, so its area is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$; remember the half, because the booklet prints only the parallelogram.

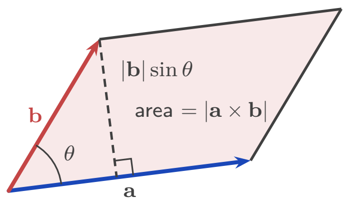


Figure 9.8 The parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. Its area is $|\mathbf{a} \times \mathbf{b}|$, and a triangle on the same two sides has half that area.

Worked through. To find the area of the triangle with vertices $A(0,0,0)$, $B(3,1,0)$, $C(1,0,2)$, take two sides from the same vertex, $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Their cross product is

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} (1)(2) - (0)(0) \\ (0)(1) - (3)(2) \\ (3)(0) - (1)(1) \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ -1 \end{pmatrix}.$$

Its magnitude is $\sqrt{4 + 36 + 1} = \sqrt{41}$, so the triangle's area is $\frac{1}{2}\sqrt{41}$. The cross product avoids trigonometry here: one calculation and one square root give the area.

YOUR TURN 9.5 The Vector Product

Calculating the vector product

25. Evaluate each vector product.

(a) $\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix}$

(b) $\begin{pmatrix}2\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\3\\0\end{pmatrix}$

(c) $\begin{pmatrix}1\\2\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\2\end{pmatrix}$

(d) $\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}4\\5\\6\end{pmatrix}$

26. Use $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$ to find each product without writing column vectors.

(a) $\hat{\mathbf{j}} \times \hat{\mathbf{k}}$

(b) $\hat{\mathbf{k}} \times \hat{\mathbf{j}}$

(c) $\hat{\mathbf{i}} \times (\hat{\mathbf{i}} + \hat{\mathbf{j}})$

(d) $(2\hat{\mathbf{i}}) \times (3\hat{\mathbf{k}})$

Perpendicular to both

27. Let $\mathbf{a} = \begin{pmatrix}2\\1\\-1\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}1\\3\\2\end{pmatrix}$.

(a) Find $\mathbf{a} \times \mathbf{b}$.

(b) Verify that $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

(c) Find a unit vector perpendicular to both.

(d) Write down $\mathbf{b} \times \mathbf{a}$, without further calculation.

Parallel vectors

28. Use the vector product to decide whether each pair of vectors is parallel.

(a) $\begin{pmatrix}2\\-1\\3\end{pmatrix}$ and $\begin{pmatrix}-4\\2\\-6\end{pmatrix}$

(b) $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}2\\4\\5\end{pmatrix}$

Areas

29. Areas and magnitudes.

(a) Given $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$ and an angle of $30°$, find $|\mathbf{a} \times \mathbf{b}|$.

(b) Find the area of the parallelogram with sides $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\2\\0\end{pmatrix}$.

(c) Find the area of the triangle with vertices $A(1, 0, 1)$, $B(3, 2, 1)$, $C(2, -1, 3)$.

(d) Explain why $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ for every vector $\mathbf{v}$.

Concept check

30. A student calculates $\begin{pmatrix}1\\2\\1\end{pmatrix} \times \begin{pmatrix}3\\0\\2\end{pmatrix} = \begin{pmatrix}4\\-1\\-6\end{pmatrix}$.

(a) Use a scalar product to show that the answer is wrong.

(b) Find the correct vector product, and state the likely error.

9.6 Planes

A plane is a flat, endless sheet, and like a line it can be built from a point and some directions. A line needed only one direction, because a line is one-dimensional: one number, the parameter, is enough to say where you are on it. A plane is two-dimensional, so it needs two directions and two numbers. Anchor at $\mathbf{a}$, then travel λ lots of $\mathbf{b}$ and μ lots of $\mathbf{c}$. The pair (λ, μ) acts as a coordinate system inside the sheet, and every point of the plane has exactly one such pair. The two directions must not be parallel: if they were, every combination of them would still point along a single line, and the points would all lie on that line.

KEY · IB **Vector equation of a plane:** $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$.

Two parameters make this form awkward to use. It is hard to tell whether a point lies on the plane, or whether two such equations describe the same plane. A better description uses not the directions the plane contains but the single direction it does *not*: the **normal vector** $\mathbf{n}$, perpendicular to the whole sheet. Every displacement within the plane is perpendicular to $\mathbf{n}$, and the scalar product states exactly that.

KEY · IB **Normal (scalar-product) form:** $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$,

Cartesian form: $n_1x + n_2y + n_3z = d$, where $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ and $d = \mathbf{a} \cdot \mathbf{n}$.

The picture behind this is simple. Take any point R on the plane. The displacement $\mathbf{r} - \mathbf{a}$ from the known point A to R lies flat in the sheet, so it is perpendicular to $\mathbf{n}$ and its scalar product with $\mathbf{n}$ is zero. Expanding $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ gives the form above.

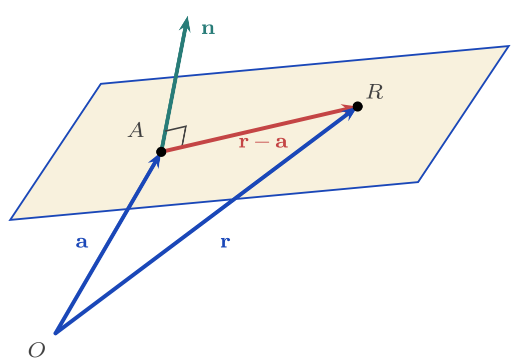


Figure 9.9 A plane held by one point and one perpendicular direction. Every displacement inside the sheet, such as $\mathbf{r} - \mathbf{a}$, is at right angles to the normal $\mathbf{n}$, so $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ for every point R of the plane, and for no other point.

A Cartesian plane equation can now be read directly: the coefficients of x, y, z *are* the normal vector. To find a plane through three points, use the cross product of two edge vectors to build $\mathbf{n}$, then fix d from any one point.

Worked through. For the plane with normal $\mathbf{n} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ through $A(4, 8, -1)$, the Cartesian equation is $3x - 2y + 2z = d$. Substitute A to find d:

$$d = 3(4) - 2(8) + 2(-1) = 12 - 16 - 2 = -6, \qquad \text{so} \qquad 3x - 2y + 2z = -6.$$

The normal supplies the coefficients and a single point supplies the constant. That is the method.

When no normal is given, the cross product produces one.

EXAMPLE 9.4

Find the Cartesian equation of the plane through $A(3, 0, 0)$, $B(0, 4, 0)$, $C(0, 0, 2)$.

Two edge vectors span the plane: $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$. Their cross product is normal to both, hence to the plane:

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} (4)(2) - (0)(0) \\ (0)(-3) - (-3)(2) \\ (-3)(0) - (4)(-3) \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 12 \end{pmatrix} = 2 \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}.$$

Any scalar multiple of a normal is a normal, so use the cleaner $(4, 3, 6)$. Through A: $d = 4(3) = 12$, giving $4x + 3y + 6z = 12$. Check with B: $3(4) = 12$; and C: $6(2) = 12$. Both lie on it. Scaling the normal down before computing d keeps every later number small.

That example settles the stool from the opening page. Any three points not in a straight line give two edge vectors; their cross product is a normal, and a normal with a point fixes a plane. So three points always have exactly one plane through them, and three feet always reach the floor. A fourth point comes with no such guarantee, so a fourth foot need not touch the floor when the other three do.

9.6.1 Angles with a plane

Two planes meet at an angle, and that angle is the angle between their normals, computed by the ordinary scalar product. To see why, look along the line where the planes meet, so that each plane is seen edge-on and appears as a line. Turning one arm of an angle through 90°, and then the other arm through 90°, leaves the angle unchanged. The normals are exactly the two arms turned in this way, so they carry the same angle.

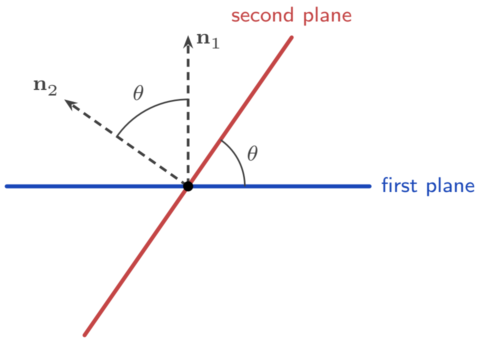


Figure 9.10 Two planes seen end-on, looking straight along the line where they meet. Each plane shows as a line. The angle between the planes and the angle between their normals are the same angle.

Taking the absolute value of the dot product guarantees the acute answer.

KEY Angle between two planes (normals $\mathbf{n}_1, \mathbf{n}_2$): $\cos\theta = \dfrac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}$.

EXAMPLE 9.5

Find the acute angle between the planes $2x + 2y - z = 4$ and $x - 2y + 2z = 1$.

Read the normals straight off the coefficients: $\mathbf{n}_1 = (2, 2, -1)$ and $\mathbf{n}_2 = (1, -2, 2)$. Then $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 4 - 2 = -4$, and both magnitudes are 3, so

$$\cos\theta = \frac{|-4|}{3 \times 3} = \frac{4}{9} \implies \theta \approx 63.6^\circ.$$

The dot product came out negative, which means the angle between the normals is obtuse. Taking the absolute value gives the acute angle instead, which is what the question asks for.

A line and a plane need one extra step. The angle we want is between the line and its projection *in* the plane: the line formed by the feet of perpendiculars dropped from the line to the plane. But the scalar product gives the angle φ between the line and the *normal*. These two angles add to 90°, so subtract from 90° to get the one we want.

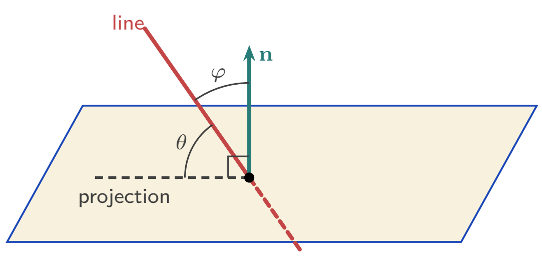


Figure 9.11 The angle between a line and a plane is measured to the line's projection in the plane, marked θ, not to the normal. Since the normal stands at 90° to the projection, the two angles at the point of entry satisfy $\theta + \varphi = 90^\circ$.

KEY Angle between a line (direction $\mathbf{d}$) and a plane (normal $\mathbf{n}$): $\sin\theta = \dfrac{|\mathbf{d}\cdot\mathbf{n}|}{|\mathbf{d}|\,|\mathbf{n}|}$.

Using $\sin$ instead of $\cos$ is what builds the "90° minus" into the formula. The line's angle to the plane is the complement of its angle to the normal.

Worked through. To find the angle between the line $\mathbf{r} = \begin{pmatrix}1\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\2\end{pmatrix}$ and the plane $2x+2y+z=4$, note that the direction is $\mathbf{d} = \begin{pmatrix}2\\1\\2\end{pmatrix}$, and the normal is read straight off the coefficients, $\mathbf{n} = \begin{pmatrix}2\\2\\1\end{pmatrix}$. Then $\mathbf{d}\cdot\mathbf{n} = 4+2+2 = 8$, with $|\mathbf{d}| = 3$ and $|\mathbf{n}| = 3$, so

$$\sin\theta = \frac{|8|}{3\times 3} = \frac{8}{9} \implies \theta = \arcsin\frac{8}{9} \approx 62.7^\circ.$$

The same fraction fed into a cosine would give 27.3°, and that is the angle to the *normal*, not to the plane. The two add to 90°, so a solution that reaches for $\arccos$ out of habit lands on the complement of the answer.

9.6.2 Intersections: line and plane, plane and plane

To find where a line meets a plane, put the line's parametric coordinates into the plane's Cartesian equation. That gives one equation in λ. Solve it, then substitute back to get the point. If the λ terms cancel, the line is parallel to the plane. The line then misses the plane when the constants disagree, and lies entirely in it when they agree.

EXAMPLE 9.6

Find where the line $\mathbf{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}$ meets the plane $3x+y-z=9$.

Parametrically, $x = 1+2\lambda$, $y = 2-\lambda$, $z = 3\lambda$. Substituting into the plane,

$$3(1+2\lambda) + (2-\lambda) - 3\lambda = 9 \implies 2\lambda + 5 = 9 \implies \lambda = 2,$$

so the point is $(5,0,6)$. Check: $3(5)+0-6 = 9$.

Two non-parallel planes do not meet at a point; they meet along a whole **line**. That line runs within both planes, so its direction is perpendicular to both normals, and the cross product builds it directly. For a point on the line, set one coordinate to zero and solve the two plane equations for the other two. If that system has no solution, the line never crosses that coordinate plane; set a different coordinate to zero instead.

Worked through. To find a vector equation of the line of intersection of the planes $x + y + 2z = 9$ and $3x - y + z = 7$, note that the direction is perpendicular to both normals:

$$\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}.$$

For a point on the line, pick a coordinate and set it to zero. With $z = 0$: $x + y = 9$ and $3x - y = 7$, so $x = 4$, $y = 5$. Hence

$$\mathbf{r} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}.$$

Check the direction: $(3, 5, -4) \cdot (1, 1, 2) = 0$ and $(3, 5, -4) \cdot (3, -1, 1) = 0$. Perpendicular to both normals, as the geometry demands.

9.6.3 Three planes

Three planes form a system of three linear equations, the same systems solved by row reduction in Ch. 4. Nothing new has to be calculated here. What is new is that every algebraic outcome now has a shape, and the shape is often easier to remember than the algebra.

There are five arrangements of three distinct planes. The first two have solutions; the last three do not, and they are told apart by how many of the normals are parallel.

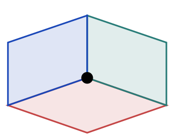
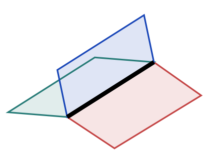
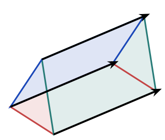
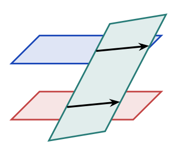
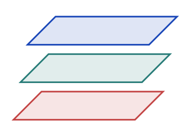
Unique solution	Infinitely many solutions	No solutions (inconsistent system)		
		No normals parallel	Two normals parallel	Three normals parallel
				
Three planes meet at a point	Three planes meet along a line	Three planes form a prism	One plane cutting two parallel planes	Three parallel planes

Figure 9.12 The five arrangements of three planes. Only the first has a single solution; the second has a whole line of them. The last three have none, and the number of parallel normals says which one you are looking at.

Three further cases arise when two or more of the equations are multiples of one another, so that they describe the same plane.

- All three describe the same plane: every point of that plane is a solution.
- Two describe the same plane and the third crosses it: the solutions form the line where they meet.
- Two describe the same plane and the third is parallel to it: there is no solution.

Row reduction tells you which case you are in without any picture. A row of zeros with a zero on the right means infinitely many solutions, so the planes share a line (or a plane, if two rows vanish). A row of zeros with a non-zero on the right means an inconsistent one, so there is no common point. The pictures only tell you what that answer looks like.

Three planes that share a common line are called a **sheaf**. To test for one, take the line where two of the planes meet and substitute its parametric coordinates into the third plane's equation. If the parameter cancels, the third plane is parallel to the line: when the constants agree it contains the line and the planes form a sheaf; when they disagree, and no two of the planes are parallel, the three planes form a prism.

Worked through. The planes $x + y + 2z = 9$ and $3x - y + z = 7$, whose line of intersection was found above, are joined by a third, $4x + 3z = c$. To describe the configuration of the three planes for $c = 16$ and for $c \neq 16$, add the first two equations. This gives $4x + 3z = 16$, so every point on their line of intersection satisfies $4x + 3z = 16$. Indeed, check the line $\mathbf{r} = (4, 5, 0) + t(3, 5, -4)$ in the third plane: $4(4 + 3t) + 3(-4t) = 16 + 12t - 12t = 16$, for every t.

So when $c = 16$ the third plane contains the entire line, and all three planes share it: they form a sheaf. When $c \neq 16$ the third plane is parallel to that line but misses it. The three planes then bound a triangular prism: each pair meets in a line, the three lines are parallel, and no point lies on all three. In the language of Ch. 4, the system has infinitely many solutions when $c = 16$ and is inconsistent otherwise; the geometry is what those words look like.

YOUR TURN 9.6 Planes

Equations of a plane

31. Equations of planes.

(a) Write the Cartesian equation of the plane with normal $\begin{pmatrix}1\\2\\3\end{pmatrix}$ through the origin.

(b) State a normal vector to the plane $2x - y + 3z = 5$.

(c) Find the Cartesian equation of the plane with normal $\begin{pmatrix}1\\1\\1\end{pmatrix}$ through $(2, 0, 1)$.

(d) Does $(1, 1, 1)$ lie on the plane $x + y + z = 3$?

32. Forms of the equation of a plane.

(a) Write the equation of the plane through $(1, -2, 3)$ with normal $\begin{pmatrix}2\\1\\-1\end{pmatrix}$ in the form $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, and in Cartesian form.

(b) A plane has vector equation $\mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\0\end{pmatrix} + \mu\begin{pmatrix}0\\1\\3\end{pmatrix}$. Find a normal vector and the Cartesian equation of the plane.

(c) Does the point $(2, 1, -2)$ lie on the plane in part (b)? Justify your answer.

33. Planes from points.

(a) Find the Cartesian equation of the plane through $A(1, 0, 2)$, $B(3, 1, 0)$ and $C(0, 2, 1)$.

(b) Find the Cartesian equation of the plane that contains the line $\mathbf{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix}$ and the point $(2, 1, 3)$.

Intersections

34. Intersections.

(a) Find where the line $\mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$ meets the plane $3x - 2y + 2z = 11$.

(b) Find a direction vector of the line of intersection of $x + y + z = 6$ and $2x - y + z = 3$.

(c) Find a point on both planes in part (b), and hence a vector equation of their line of intersection.

Angles with a plane

35. GDC Find each acute angle.

(a) The angle between the planes $x + 2y - z = 3$ and $2x - y + z = 1$.

(b) The angle between the planes $x + y = 3$ and $y + z = 1$.

(c) The angle between the line $\mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$ and the plane $3x - 2y + 2z = 11$.

(d) The angle between the line $\mathbf{r} = \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$ and the plane $z = 0$.

Three planes

36. For each system of three planes, decide whether they meet at a single point, in a line, or not at all, and describe the arrangement (for example, a sheaf or a triangular prism). Give the point or the line where there is one.

(a) $x + y + z = 6, \quad x - y + z = 2, \quad 2x + y - z = 1$
(b) $x + y + z = 6, \quad x - y + z = 2, \quad 2x + 2z = 8$
(c) $x + y + z = 6, \quad 2x + 2y + 2z = 10, \quad x - y + z = 2$
(d) $x + y + z = 6, \quad x - y + z = 2, \quad 2x + 2z = 3$

Concept check

37. GDC To find the angle between the line $\mathbf{r} = \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and the plane $z = 0$, a student uses the direction vector and the normal $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in $\cos\theta = \dfrac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}|\,|\mathbf{n}|}$, and gets $\theta \approx 48.2°$. Identify the error and find the correct angle.

9.7 Distances

Two objects that never meet are still some distance apart, and asking for that distance always means the same thing: the *shortest* one. There is a single idea behind every case in this section, and it needs no new formula.

Shortest distance is not itself listed in the syllabus, which asks for intersections and angles. Every step below is built from tools that are listed, and exam questions can reach the same answers by asking for the foot of a perpendicular. Treat this section as the natural extension of what comes before, not as separate material.

> KEY **The shortest link is perpendicular.** Of all the segments joining two objects, the shortest is the one at right angles to what it joins. Perpendicular means a scalar product of zero, so the geometry turns straight into an equation.

That gives a method rather than a formula to memorise. Nothing here is printed in the formula booklet, so working it out is the point.

KEY **Finding a shortest distance.**

1. Write the unknown point on each object in terms of its parameter.
2. Set the scalar product of the joining vector with each direction it must be perpendicular to equal to zero.
3. Solve for the parameter or parameters, which locates the feet of the perpendicular.
4. Find the length of the joining vector.

9.7.1 From a point to a line

A general point on the line is $F = \mathbf{a} + \lambda\mathbf{b}$. As λ varies, F slides along the line and the length PF changes. It is shortest exactly when $\overrightarrow{PF}$ is perpendicular to the line, so $\overrightarrow{PF} \cdot \mathbf{b} = 0$. That is one equation in one unknown.

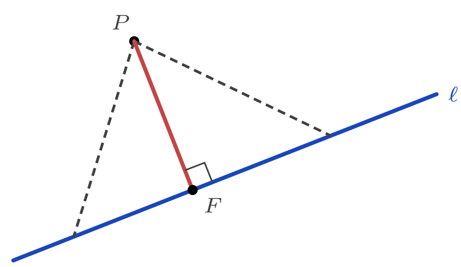


Figure 9.13 Every dashed segment joins P to the line. The shortest of them meets the line at right angles, at the foot of the perpendicular F.

Worked through. To find the distance from $P(0, 1, 3)$ to the line $\mathbf{r} = \begin{pmatrix}1\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$, write a general point on the line as $F = (1 + \lambda,\ 2\lambda,\ -1 + 2\lambda)$, so

$$\overrightarrow{PF} = \begin{pmatrix}1+\lambda\\2\lambda-1\\2\lambda-4\end{pmatrix}.$$

Now make it perpendicular to the direction:

$$\overrightarrow{PF} \cdot \begin{pmatrix}1\\2\\2\end{pmatrix} = (1+\lambda) + 2(2\lambda-1) + 2(2\lambda-4) = 9\lambda - 9 = 0,$$

so $\lambda = 1$ and the foot is $F(2, 2, 1)$. Then $\overrightarrow{PF} = (2, 1, -2)$ and the distance is $\sqrt{4+1+4} = 3$. Notice what the method gives you for free: not just the distance but the point on the line that is closest to P, which many questions ask for as well.

9.7.2 From a point to a plane

For a plane the perpendicular direction is already known: it is the normal $\mathbf{n}$. So no scalar product needs to be set to zero. Instead, walk from P along $\mathbf{n}$ until you land on the plane. The general point on that walk is $\mathbf{p} + t\mathbf{n}$, and putting it into $\mathbf{r} \cdot \mathbf{n} = d$ gives one equation in t. The distance travelled is then $|t|\,|\mathbf{n}|$.

Worked through. To find the distance from $P(1, 1, 1)$ to the plane $2x - y + 2z = 12$, note that the normal is $\mathbf{n} = (2, -1, 2)$, so a point on the walk from P is

$$\mathbf{p} + t\mathbf{n} = (1 + 2t,\ 1 - t,\ 1 + 2t).$$

This lies on the plane when

$$2(1 + 2t) - (1 - t) + 2(1 + 2t) = 12 \implies 9t + 3 = 12 \implies t = 1.$$

The foot is $(3, 0, 3)$, and the distance is $|t|\,|\mathbf{n}| = 1 \times 3 = 3$. Check the foot on the plane: $6 - 0 + 6 = 12$. ✓

9.7.3 Between two skew lines

Skew lines never meet, yet there is still a shortest gap between them, and the same idea finds it. Take a general point on each, P_1 on the first and P_2 on the second. The joining vector $\overrightarrow{P_1P_2}$ is shortest when it is perpendicular to *both* directions at once. That is two scalar products set to zero, so two equations in the two unknowns λ and μ.

Worked through. To find the shortest distance between $l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, write a general point on each line: $P_1 = (\lambda,\ \lambda,\ 1 + \lambda)$ and $P_2 = (1 + \mu,\ 1,\ \mu)$, so

$$\overrightarrow{P_1P_2} = \begin{pmatrix} 1 + \mu - \lambda \\ 1 - \lambda \\ \mu - 1 - \lambda \end{pmatrix}.$$

Perpendicular to both directions gives two equations:

$$\overrightarrow{P_1P_2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 - 3\lambda + 2\mu = 0, \qquad \overrightarrow{P_1P_2} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2\mu - 2\lambda = 0.$$

The second gives $\mu = \lambda$; the first then gives $1 - \lambda = 0$, so $\lambda = \mu = 1$. The feet are

$P_1(1,1,2)$ and $P_2(2,1,1)$, and

$$\overrightarrow{P_1P_2} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \text{distance} = \sqrt{2}.$$

Two ordinary simultaneous equations, and the answer comes with both endpoints of the shortest segment attached.

9.7.4 The remaining cases

Nothing else needs a new method. Every other pairing either meets, and so has distance zero, or is parallel, and then any convenient point reduces it to one of the three cases above.

The two objects	What to do
Two parallel lines	Take any point on one line, then use point to line
Two intersecting lines	They meet, so the distance is 0
Two skew lines	Two scalar products set to zero, as above
A line parallel to a plane	Take any point on the line, then use point to plane
A line meeting a plane	They meet, so the distance is 0
Two parallel planes	Take any point on one plane, then use point to plane
Two non-parallel planes	They meet along a line, so the distance is 0

The first step in any of these questions is therefore not calculation but classification. Decide whether the objects meet, and if they do not, decide which of the three methods the pair reduces to.

EXAMPLE 9.7

Find the distance between the parallel planes $2x - y + 2z = 12$ and $2x - y + 2z = 3$.

Both have normal $(2, -1, 2)$, so they are parallel and never meet. Take any point on the second plane; setting $y = z = 0$ gives $x = \frac{3}{2}$, so $A(\frac{3}{2}, 0, 0)$ lies on it. Now walk from A along the normal to the first plane:

$$2\left(\tfrac{3}{2} + 2t\right) - (-t) + 2(2t) = 12 \implies 9t + 3 = 12 \implies t = 1,$$

so the distance is $|t|\,|\mathbf{n}| = 3$. Any other starting point on the second plane gives the same answer, which is what parallel means.

YOUR TURN 9.7 Distances

From a point to a line

38. Distance from a point to a line. Throughout, $l : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

(a) Find the foot of the perpendicular from $P(6, 2, 1)$ to l, and hence the distance from P to l.

(b) Find the distance from $Q(3, 3, 0)$ to l.

(c) The line m has equation $\mathbf{r} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -4 \\ 2 \\ -4 \end{pmatrix}$. Explain why m is parallel to l, and write down the distance between them.

(d) Check your answer to part (c) by repeating the method with the point on m where $t = 1$.

From a point to a plane

39. Distance from a point to a plane. Throughout, $\Pi : x + 2y + 2z = 14$.

(a) Find the foot of the perpendicular from $P(0, -1, -1)$ to Π, and hence the distance from P to Π.

(b) Find the distance from the origin to Π.

(c) Find the distance between Π and the parallel plane $x + 2y + 2z = 5$.

(d) Show that the line $\mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ is parallel to Π, and write down the distance between the line and the plane.

Between two lines

40. Distance between two lines. Throughout, $l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

(a) Show that l_1 and l_2 are skew.

(b) Write down general points P_1 on l_1 and P_2 on l_2, and find the values of λ and μ for which $\overrightarrow{P_1P_2}$ is perpendicular to both direction vectors.

(c) Hence state the two feet of the common perpendicular and the shortest distance between l_1 and l_2.

(d) The lines $n_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $n_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are not parallel. Apply the same method to them and explain what your answer shows.

Concept check

41. A student says: "*the distance from the origin to the plane $x + 2y + 2z = 14$ is 14.*" Identify the error and find the correct distance.

Chapter 9 · Vectors

Reflections

TOK The same plane can be a drawing, a vector equation, or the single equation $2x - y + 3z = 7$. The symbols are easier to calculate with. The picture is easier to believe.

If the two disagree in your head, which one do you trust, and what happens in four dimensions, where the picture is no longer available to you?

IM The products of this chapter were chosen, not found. On 16 October 1843 the Irish mathematician William Rowan Hamilton, walking beside a canal in Dublin, saw how to multiply numbers of the form $a + bi + cj + dk$, and carved the rule $i^2 = j^2 = k^2 = ijk = -1$ into the stone of Brougham Bridge. His *quaternions* multiply in a way that depends on the order, as the cross product does. About forty years later the American Josiah Willard Gibbs and the English engineer Oliver Heaviside cut quaternions down to the two operations physicists actually used, the dot product and the cross product, and their vector notation replaced Hamilton's. What should decide which of two definitions survives: which is more elegant, or which is easier to use?

RW Three-dimensional games and animated films rely on this chapter. The dot product decides how brightly a surface is lit, by comparing the direction of the light with the direction the surface faces: a value near 1 points at the light and is bright, a value near 0 faces edge-on and stays dark. The cross product builds the normals that say which way each surface points in the first place. At the same time the physics engine adds force and velocity vectors tip to tail, many times a second, so a thrown object curves the way your eye expects.

EXT The cross product, as defined here, works only in three dimensions.

In two dimensions there is no room for a vector perpendicular to two others. In four dimensions or more there is too much room: no single perpendicular direction is picked out, because a whole family of them works. Three dimensions is the only case where two vectors determine exactly one perpendicular line. That is why $\mathbf{a} \times \mathbf{b}$ can be a vector at all. (A product with most of the same properties also exists in seven dimensions, and in no others.)

End-of-Chapter Problems

1. Given $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$:

(a) find $\mathbf{a} + 2\mathbf{b}$; [2]

(b) find $|\mathbf{a}|$; [1]

(c) find the unit vector in the direction of $\mathbf{a}$. [2]

2. For $A(1, -1, 2)$ and $B(3, 2, 4)$:

(a) find $\overrightarrow{AB}$; [1]

(b) find $|\overrightarrow{AB}|$; [2]

(c) find the midpoint of AB. [2]

3. The point $M(3, -1, 4)$ is the midpoint of AB, where A is $(1, 2, 5)$. Find the coordinates of B. [3]

4. Show that the points $A(1, 2, 3)$, $B(2, 4, 5)$ and $C(4, 8, 9)$ are collinear. [3]

5. The vectors $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ k \end{pmatrix}$ are perpendicular. Find k. [3]

6. Find the angle between $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. [4]

7. GDC Given $|\mathbf{a}| = 5$, $|\mathbf{b}| = 3$ and $\mathbf{a} \cdot \mathbf{b} = 9$, find the angle between $\mathbf{a}$ and $\mathbf{b}$. [3]

8. GDC Find, in degrees, the acute angle between two lines with direction vectors $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$. [3]

9. Given $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, find $\mathbf{a} \times \mathbf{b}$ and $|\mathbf{a} \times \mathbf{b}|$. [4]

10. Find a vector equation of the line through $A(2, -1, 3)$ and $B(4, 1, 7)$. [3]

11. Consider the line $l : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

(a) Find the point on l when $\lambda = 2$. [2]

(b) Determine whether $P(7, 3, 4)$ lies on l. [2]

12. The line $l : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ meets the plane $z = 4$. Find the point of intersection. [3]

13. A plane Π has normal $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and passes through $A(1,2,-1)$.

(a) Find the Cartesian equation of Π. [3]

(b) Determine whether $B(2,1,0)$ lies on Π. [1]

14. The plane Π has equation $2x+y-2z=6$.

(a) State a normal vector to Π. [1]

(b) Find where the line $\mathbf{r} = \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ meets Π. [4]

15. The lines $l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ intersect. Find the point of intersection. [5]

16. GDC Find the acute angle between the planes $x+y+z=1$ and $x-y+z=3$. [4]

17. Find a unit vector perpendicular to both $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. [5]

18. The points $A(1,0,0)$, $B(0,2,0)$ and $C(0,0,3)$ are given.

(a) Find the Cartesian equation of the plane ABC. [3]

(b) Find the area of triangle ABC. [3]

19. Find a vector equation of the line of intersection of the planes $x+y-z=2$ and $2x-y+z=1$. [6]

20. Show that the lines $l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ intersect, and find the point of intersection. [6]

21. The point P is $(1,2,3)$ and the plane Π has equation $2x-y+2z=0$. Find the coordinates of the foot of the perpendicular from P to Π, and hence the shortest distance from P to Π. [6]

22. The line l has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, and P is the point $(4,3,2)$. The point F lies on l.

(a) Write down $\overrightarrow{PF}$ in terms of λ. [2]

(b) Find the value of λ for which $\overrightarrow{PF}$ is perpendicular to l, and hence the coordinates of the foot of the perpendicular from P to l. [3]

(c) Hence find the shortest distance from P to l. [2]

23. GDC The points $A(1,2,-1)$, $B(4,0,3)$ and $C(-2,5,1)$ are the vertices of a triangle.

(a) Find $|\overrightarrow{AB}|$ and $|\overrightarrow{AC}|$. [2]

(b) Find the angle $B\hat{A}C$. [3]

(c) Find the area of triangle ABC, and hence the shortest distance from C to the line AB. [3]

24. GDC Find the angle that the line $l : \mathbf{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$ makes with the plane $x + 2y + 2z = 5$. [5]

25. At time $t = 0$ seconds a drone is at the point $(2, -1, 10)$ (coordinates in metres) and moves with constant velocity $\begin{pmatrix}3\\4\\0\end{pmatrix}$ m s^{-1}.

(a) Write down a vector equation for the drone's position at time t. [2]

(b) Find the drone's speed. [2]

(c) Determine whether the drone passes through the point $(17, 19, 10)$, and if so, at what time. [3]

26. Consider the planes $\Pi_1 : x + y + z = 3$, $\Pi_2 : x - y + z = 1$ and $\Pi_3 : x + z = c$, where $c \in \mathbb{R}$.

(a) Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

(b) Show that when $c = 2$ the three planes intersect in this line. [2]

(c) Describe geometrically the configuration of the three planes when $c \neq 2$. [2]

27. The points $A(1, 2, 1)$, $B(3, 1, 2)$ and $C(2, 4, 3)$ are given.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [2]

(b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [3]

(c) Hence find the Cartesian equation of the plane Π through A, B and C. [3]

(d) The line l has equation $\mathbf{r} = \begin{pmatrix}0\\0\\1\end{pmatrix} + t\begin{pmatrix}1\\1\\1\end{pmatrix}$. Find the coordinates of the point where l meets Π. [3]

(e) The line m has equation $\mathbf{r} = \begin{pmatrix}0\\0\\1\end{pmatrix} + s\begin{pmatrix}1\\k\\1\end{pmatrix}$. Find the value of k for which m is parallel to Π. [3]

28. The points $A(2, 0, 1)$, $B(1, 3, 2)$ and $C(4, 1, 0)$ are given, and O is the origin.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [2]

(b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [3]

(c) Find the exact area of triangle ABC. [2]

(d) Find the Cartesian equation of the plane through A, B and C. [4]

(e) Find the exact shortest distance from O to that plane. [3]

(f) Hence find the volume of the tetrahedron $OABC$. [2]

29. Consider the lines

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

(a) Show that l_1 and l_2 are not parallel. [2]
(b) Determine whether the lines intersect, and if so find the point of intersection. [5]
(c) Find the angle between the lines. [3]
(d) Find the Cartesian equation of the plane containing both lines. [4]
(e) Explain why such a plane exists only because of your answer to part (b). [2]

30. The lines l_1 and l_2 have equations

$$l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

(a) Show that l_1 and l_2 are not parallel. [2]
(b) Show that l_1 and l_2 do not intersect. [4]
(c) Find $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. [2]
(d) The point P on l_1 has parameter s, and the point Q on l_2 has parameter t. Find $\overrightarrow{PQ}$ in terms of s and t. [2]
(e) The line PQ is perpendicular to both l_1 and l_2. Show that $s = 1$ and $t = 1$, and find the coordinates of P and Q. [4]
(f) Hence find the shortest distance between l_1 and l_2, and verify that $\overrightarrow{PQ}$ is parallel to your answer to part (c). [2]

31. GDC Two aircraft fly at the same altitude. At time t hours their position vectors, in km, are

$$\mathbf{r}_A = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 300 \\ 400 \\ 0 \end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix} 600 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 500 \\ 0 \end{pmatrix}.$$

(a) Find the speed of each aircraft. [3]
(b) Find $\overrightarrow{AB}$ at time t. [3]
(c) Show that the distance d between them satisfies $d^2 = 100\,000t^2 - 360\,000t + 360\,000$. [4]
(d) Find the time at which the aircraft are closest, and the least distance between them. [4]

32. Consider the planes

$$\Pi_1 : x + y + z = 1, \qquad \Pi_2 : x + 2y + 3z = 4, \qquad \Pi_3 : x + y + kz = m,$$

where k and m are real constants.

(a) Find a vector equation of the line of intersection of Π_1 and Π_2. [4]
(b) Show that when $k = 2$ the three planes meet at a single point, and find that point in terms of m. [4]
(c) Find the values of k and m for which the three planes meet in a line. [3]
(d) Find the values of k and m for which the three planes have no common point, and

describe that arrangement geometrically. [3]

33. GDC Two drones move with constant velocity. At time t seconds their positions, in metres, are

$$\mathbf{r}_A = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix} 3 \\ -13 \\ 6 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}.$$

(a) Find the speed of each drone. [3]

(b) Show that the two flight paths cross, and find the point where they cross. [5]

(c) Explain why the drones nevertheless do not collide. [3]

(d) Find the distance between the drones at $t = 2$. [3]

(e) Find the acute angle between the two flight paths. [3]

Challenge Questions

34. **Closest approach of two moving objects.** Two drones fly in straight lines at constant velocity. Taking $t \geq 0$ in seconds and distances in metres, their positions are

$$\mathbf{r}_A = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

Section 9.4 finds a closest approach by completing the square for one pair of objects. This investigation turns the method into a formula, links it to the vector product, and compares it with the distance between the two flight paths.

In general write $\mathbf{r}_A = \mathbf{a}_A + t\mathbf{v}_A$ and $\mathbf{r}_B = \mathbf{a}_B + t\mathbf{v}_B$, with $\Delta\mathbf{a} = \mathbf{a}_B - \mathbf{a}_A$ and $\Delta\mathbf{v} = \mathbf{v}_B - \mathbf{v}_A \neq \mathbf{0}$, and let d be the distance between the two objects at time t.

(a) For the drones, show that $d^2 = 4(t-2)^2 + 4$. State the time of closest approach and the least distance. [4]

(b) In general, show that $d^2 = |\Delta\mathbf{v}|^2 t^2 + 2(\Delta\mathbf{a}\cdot\Delta\mathbf{v})\,t + |\Delta\mathbf{a}|^2$. By completing the square, show that d^2 is least at

$$t^* = -\frac{\Delta\mathbf{a}\cdot\Delta\mathbf{v}}{|\Delta\mathbf{v}|^2}, \qquad \text{with} \qquad d_{\min}^2 = |\Delta\mathbf{a}|^2 - \frac{(\Delta\mathbf{a}\cdot\Delta\mathbf{v})^2}{|\Delta\mathbf{v}|^2},$$

and state what it means for the two objects when $t^* < 0$. [5]

(c) Use $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$ and $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$ to show that $|\mathbf{u}\times\mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\cdot\mathbf{v})^2$. Hence show that $d_{\min} = \dfrac{|\Delta\mathbf{a}\times\Delta\mathbf{v}|}{|\Delta\mathbf{v}|}$, and check this against part (a). [4]

(d) The two flight paths are lines. Use the method of Section 9.7 to find the two points where the common perpendicular of the paths meets them, and show that the shortest distance between the paths is $\sqrt{2}$ metres. [4]

(e) Explain why the least distance between two moving objects can never be less than the distance between their paths. Find the times at which A and B pass the two points found in part (d), and use them to explain why the drones never come as close as $\sqrt{2}$ metres. State when the two distances are equal. [3]

(f) Suppose drone B starts s seconds late, so that $\mathbf{r}_B = \mathbf{a}_B + (t-s)\mathbf{v}_B$, where a negative s means an early start. Investigate how the least distance between the drones depends on s, and suggest how this model of a near miss could be taken further. [3]

35. How far apart are two skew lines? Two skew lines never meet, yet there is still a shortest distance between them. This investigation finds it, and then deals with the case the formula cannot handle.

Let $l_1 : \mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $l_2 : \mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$ be skew.

(a) Show that $l_1 : \mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $l_2 : \mathbf{r} = \begin{pmatrix}2\\3\\1\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\2\end{pmatrix}$ are skew. [4]

(b) The shortest join between the two lines is perpendicular to both. Explain why its direction must be parallel to $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$. [3]

(c) Hence explain why the shortest distance is the length of the projection of $\mathbf{a}_2 - \mathbf{a}_1$ onto $\mathbf{n}$, that is

$$\text{distance} = \frac{|(\mathbf{a}_2 - \mathbf{a}_1)\cdot\mathbf{n}|}{|\mathbf{n}|}.$$

(Hint: let the shortest join run from P_1 on l_1 to P_2 on l_2, and write $\mathbf{a}_2 - \mathbf{a}_1$ in terms of $\overrightarrow{P_1P_2}$, $\mathbf{d}_1$ and $\mathbf{d}_2$) [4]

(d) Use the formula on the two lines in part (a), giving an exact answer. [5]

(e) What does the formula give when the two lines intersect? Explain why your answer is what it should be. [3]

(f) Now let $l_3 : \mathbf{r} = \begin{pmatrix}3\\1\\0\end{pmatrix} + \mu\begin{pmatrix}2\\2\\2\end{pmatrix}$, which is parallel to l_1. Explain why the formula of part (c) cannot be used for l_1 and l_3. Show that the distance between two parallel lines is $\dfrac{|(\mathbf{a}_3 - \mathbf{a}_1)\times\mathbf{d}_1|}{|\mathbf{d}_1|}$, and find the exact distance between l_1 and l_3. [4]

(g) The lines here are infinite. Suggest how the method would have to change to find the shortest distance between two line segments, such as two rods or two edges of a solid, and how this could be investigated further. [2]

36. The triple product and volume. Combining both products gives $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$, a single number called the *scalar triple product*. It measures volume, and it decides whether three planes meet in a single point.

In parts (a) to (e), take $\mathbf{u} = \begin{pmatrix}1\\2\\0\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}0\\1\\2\end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix}2\\0\\1\end{pmatrix}$.

(a) Find $\mathbf{v}\times\mathbf{w}$, then $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$. [3]

(b) The parallelepiped with edges $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ has base area $|\mathbf{v}\times\mathbf{w}|$. Explain why its volume is $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$. *(Hint: volume = base area × perpendicular height)* [3]

(c) Evaluate $\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})$, $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})$ and $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v})$. Make a conjecture about what happens to the triple product when the three vectors are moved round in order, and when two of them are swapped. Prove both parts of your conjecture for any three vectors. [5]

(d) Three vectors are *coplanar* when they all lie in one plane. Prove that three vectors are coplanar if and only if their triple product is zero, proving both directions. Test $\begin{pmatrix}1\\2\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\2\end{pmatrix}$, $\begin{pmatrix}2\\5\\2\end{pmatrix}$. [5]

(e) A tetrahedron on the same three edges has one sixth of the parallelepiped's volume. Find the volume of the tetrahedron with vertices at the origin and at the three points $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ above. [2]

(f) Three planes have normals $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$. Explain why they meet in exactly one point when $\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) \neq 0$, and why they cannot meet in exactly one point when it is zero. Use this to show that the planes $x + 2y = 3$, $y + 2z = 1$ and $2x + 5y + 2z = k$ never meet in a single point, and find the value of k for which they share a line. [5]

(g) The sign of the triple product has been set aside so far. Explain what the sign of $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ tells you about the three vectors, and suggest how this could be investigated further. [2]

37. Varignon's theorem in three dimensions. Varignon's theorem states that joining the midpoints of the sides of a quadrilateral in a plane, in order, gives a parallelogram. This investigation asks whether the four corners need to lie in a plane at all.

Let A, B, C, D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and let P, Q, R, S be the midpoints of AB, BC, CD and DA.

(a) Take $A(0,0,0)$, $B(4,0,0)$, $C(6,4,2)$ and $D(0,2,6)$. Show that these four points do not lie in one plane. *(Hint: use the triple product of Challenge Question 36)* [2]

(b) For the points in part (a), find P, Q, R and S, and show that $PQRS$ is a parallelogram with four equal sides. [3]

(c) Prove that for any four points $\overrightarrow{PQ} = \overrightarrow{SR} = \frac{1}{2}(\mathbf{c} - \mathbf{a})$ and $\overrightarrow{QR} = \overrightarrow{PS} = \frac{1}{2}(\mathbf{d} - \mathbf{b})$. Deduce that $PQRS$ is always a parallelogram, lying in one plane even when A, B, C, D do not. [3]

(d) Show that the diagonals PR and QS bisect each other at the point G with position vector $\frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$, and that G is also the midpoint of the segment joining the midpoints of AC and BD. Find G for the points in part (a). [3]

(e) Compare the lengths of AC and BD in part (a). Conjecture a condition on AC and BD for $PQRS$ to be a rhombus, and another for it to be a rectangle, and prove both. Hence state what shape $PQRS$ is when $ABCD$ are the vertices of a regular tetrahedron, justifying your answer. [5]

(f) Show that the area of $PQRS$ is $\frac{1}{4}|(\mathbf{c} - \mathbf{a}) \times (\mathbf{d} - \mathbf{b})|$, and find it for the points in part (a). Compare it with the area of $ABCD$ when $ABCD$ is a convex quadrilateral in a plane, and suggest how this could be investigated further. [3]