GEOMETRY & TRIGONOMETRY

CHAPTER 8

Trigonometry

8

SYLLABUS 3.5–3.11

Your voice is a wave. So is the light from your screen, the signal carrying this page to your device, the current in the wall socket, the tide in Tokyo Bay, and the beating of your own heart.

They do not look alike. A heartbeat is a sharp spike, a spoken vowel is jagged, a tide is a slow rise and fall over twelve hours. In 1822 Joseph Fourier claimed that a repeating signal, however jagged, can be built up from pure sine waves; Peter Dirichlet later proved it for signals that are not too irregular. Choose enough of them, with the right heights and the right timings, and the shape appears. Phone audio compression uses this idea many times a second.

So all of them can be built, at least approximately, from one curve. Where does it come from?

Watch one seat on a turning Ferris wheel. As the wheel turns steadily, the seat rises, slows near the top, falls, slows near the bottom, and rises again. Plot its height against time and the sine curve appears. The seat is a point moving round a circle, and its height is the vertical coordinate of that point. The circle and the wave are the same motion seen two ways: one going round, one going along.

That is why this chapter starts with a circle. In Ch. 7 sine and cosine were defined as ratios in a right-angled triangle, which only makes sense for acute angles, even though the sine and cosine rules already used obtuse ones. On the circle they become functions of *any* angle, and they repeat forever.

By the end of this chapter you will be able to read exact values of sine, cosine and tangent from the unit circle. You will use identities to rewrite one expression as another, sketch and transform the graphs of these functions, and work with the reciprocal and inverse functions. You will also solve trigonometric equations, where a single value of the sine or cosine is shared by two different angles, so answers usually have to be looked for in pairs.

8.1 The Unit Circle

How do you find $\sin 120°$? Or $\cos 360°$, or $\tan(-30°)$?

You already have a definition, and it is SOHCAHTOA. In a right triangle,

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$

Now try to draw the triangle. For $120°$ there is none: the three angles add to $180°$, one of them is already the right angle, so the other two must each be under $90°$. For $360°$ there is none either, and $-30°$ is worse, because a triangle has no negative angles. Under SOHCAHTOA these three expressions have no meaning at all.

Yet your calculator answers all three without complaint, and a turn of $120°$ is an ordinary thing to make. So the definition has to widen. Whatever replaces it must agree with SOHCAHTOA wherever SOHCAHTOA works, and keep working where it does not.

The unit circle does exactly that. Take the circle of radius 1 centred at the origin, and a point P on it at angle θ, measured anticlockwise from the positive x-axis.

For an acute θ, drop a perpendicular from P to the x-axis. This makes a right triangle whose hypotenuse is the radius, so the hypotenuse is 1. The old ratios then read

$$\sin\theta = \frac{\text{opposite}}{1} = \text{opposite}, \qquad \cos\theta = \frac{\text{adjacent}}{1} = \text{adjacent},$$

and those two sides are exactly the coordinates of P. So for acute angles the new definition agrees with the old one.

The new form never mentions a triangle. A point can sit anywhere on the circle, at any angle, and it always has coordinates. That is what extends the definition to every real θ.

DEF On the unit circle, for a point P at angle θ:

$$\cos\theta = x\text{-coordinate of } P, \qquad \sin\theta = y\text{-coordinate of } P.$$

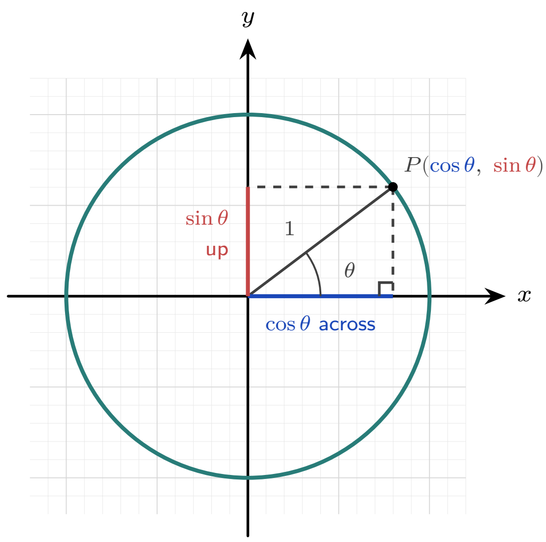

Figure 8.1 On a circle of radius 1 the hypotenuse is 1, so the two shorter sides *are* the coordinates of P. The cobalt side, measured across, is $\cos\theta$; the scarlet side, measured up, is $\sin\theta$.

The tangent is the gradient of the line OP, so $\tan\theta = \frac{\text{rise}}{\text{run}} = \frac{\sin\theta}{\cos\theta}$. It is undefined wherever $\cos\theta = 0$.

> KEY · IB $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

That word *gradient* is not accidental. The line OP passes through the origin, and its gradient is $\tan\theta$, so its equation is $y = x\tan\theta$. This ties the tangent to the straight lines of Ch. 4: the tangent converts an angle into a gradient, and $\arctan$ converts back. A line of gradient 1 makes an angle of $\frac{\pi}{4}$ with the positive x-axis, because $\tan\frac{\pi}{4} = 1$.

> KEY A line through the origin making an angle θ with the positive x-axis has equation
>
> $$y = x\tan\theta.$$
>
> Read forwards it gives the gradient from the angle; read backwards, $\theta = \arctan(\text{gradient})$ gives the angle from the gradient.

8.1.1 Exact values

A few angles give exact coordinates, and they come from the 30–60–90 and 45–45–90 triangles.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Only three numbers appear in the shaded row, and they are all that has to be memorised:

$$\frac{1}{2}, \qquad \frac{\sqrt{2}}{2}, \qquad \frac{\sqrt{3}}{2}.$$

The cosine row is those same three in reverse order. That is not a coincidence. Since $\cos\theta = \sin(\frac{\pi}{2} - \theta)$, reading the sine row backwards reads the cosine row forwards. The tangent row needs nothing new at all: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, so divide one row by the other. At $\theta = \frac{\pi}{2}$ the cosine is zero, which is why the tangent has no value there.

The first-quadrant values repeat into the other three quadrants with signs changed. Collecting them all onto one circle gives the diagram to learn.

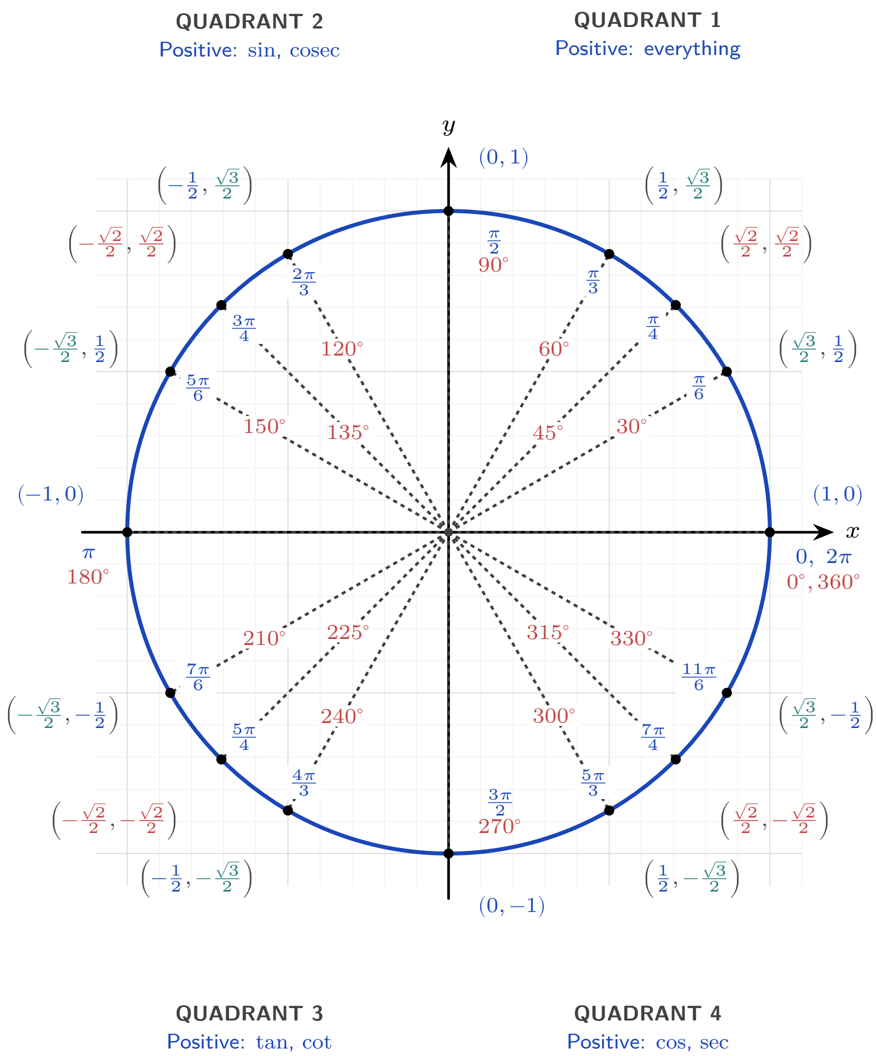

Figure 8.2 The unit circle with every special angle. Each spoke carries the angle twice: in cobalt on the outer ring in radians, and in scarlet on the inner ring in degrees. Outside the circle is the point $(\cos\theta,\ \sin\theta)$, with each coordinate coloured by its size: cobalt for $\frac{1}{2}$, scarlet for $\frac{\sqrt{2}}{2}$, teal for $\frac{\sqrt{3}}{2}$. To read a value, find the angle and take the matching coordinate. The corner labels say which functions are positive in each quadrant; $\sec$, cosec and $\cot$ are the reciprocal functions of §8.4.

The colours make the point. Only three numbers appear anywhere on the circle, and every coordinate on it is one of those three with a sign attached. Add the four points on the axes and the diagram is complete.

8.1.2 Symmetry and the sign of each function

Since $\cos\theta$ and $\sin\theta$ are the coordinates of P, their signs are just the signs of x and y in each quadrant. Nothing needs to be memorised separately: in the second quadrant x is negative and y is positive, so $\cos\theta < 0$ and $\sin\theta > 0$, and $\tan\theta = \frac{\sin}{\cos}$ is negative.

The unit-circle figure above records the answer at each corner. Reading the four quadrants anticlockwise from the first, the functions that stay positive are all three, then sine alone, then tangent alone, then cosine alone.

The mnemonic **All Students Take Calculus** is the usual way to remember that order. Use it as a check, not as the reason. The reason is always the sign of the coordinates, and if you can picture the point you never need the letters at all.

Each symmetry of the circle now gives an identity, and each one is a statement about where a reflected point lands.

Reflect in the y-axis. The point at angle $\pi - \theta$ is the mirror image, across the y-axis, of the point at angle θ. Reflecting in the y-axis negates the x-coordinate and leaves y alone, so the two points are $(-\cos\theta,\ \sin\theta)$ and $(\cos\theta,\ \sin\theta)$. Comparing coordinates:

$$\sin(\pi - \theta) = \sin\theta, \qquad \cos(\pi - \theta) = -\cos\theta.$$

Reflect in the x-axis. Measuring $-\theta$ means turning clockwise instead of anticlockwise, which lands on the mirror image of P across the x-axis. That reflection negates y and leaves x alone, so

$$\cos(-\theta) = \cos\theta, \qquad \sin(-\theta) = -\sin\theta.$$

Turn half a circle. Adding π sends P to the diametrically opposite point, which negates both coordinates:

$$\sin(\theta + \pi) = -\sin\theta, \qquad \cos(\theta + \pi) = -\cos\theta.$$

Turn a full circle. Adding 2π to the angle sends P once round and back to exactly the same point, so both coordinates are unchanged:

$$\sin(\theta + 2\pi) = \sin\theta, \qquad \cos(\theta + 2\pi) = \cos\theta.$$

This is why both functions are **periodic** with period 2π. Every identity of this kind says the same thing: move the point, see which coordinate changes sign.

You have met the first of these relations before. Two angles, θ and $\pi - \theta$, share the same sine. That shared value is what produced the ambiguous case of the sine rule in Ch. 7. A triangle built from a sine value can sometimes close in two ways, one acute and one obtuse.

An exact value at any special angle now comes out of the circle in two moves: find the acute angle the arm makes with the x-axis, which fixes the size, then look at the quadrant, which fixes the sign.

Worked through. To find $\sin\frac{5\pi}{3}$ and $\cos\frac{5\pi}{3}$ exactly, write the angle as $2\pi - \frac{\pi}{3}$. A full turn returns to the positive x-axis, so turning back through $\frac{\pi}{3}$ leaves the point in the fourth quadrant, where the x-coordinate is positive and the y-coordinate negative. The arm makes an acute angle of $\frac{\pi}{3}$ with the axis, so both values have the size they have at $\frac{\pi}{3}$:

$$\sin\frac{5\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}, \qquad \cos\frac{5\pi}{3} = \cos\frac{\pi}{3} = \frac{1}{2}.$$

Dividing them gives $\tan\frac{5\pi}{3} = -\sqrt{3}$, negative, because only the sine carries a minus sign.

That is the fourth quadrant's entry in the mnemonic, recovered from the coordinates rather than remembered.

YOUR TURN 8.1 The Unit Circle

Exact values

1. Write down the exact value of each.

(a) $\sin\frac{\pi}{6}$ (b) $\cos\frac{\pi}{4}$
(c) $\tan\frac{\pi}{3}$ (d) $\tan\frac{\pi}{4}$

2. Use the symmetry of the circle to find each exact value.

(a) $\sin\frac{5\pi}{6}$ (b) $\cos\frac{2\pi}{3}$
(c) $\sin\frac{4\pi}{3}$ (d) $\cos\frac{7\pi}{4}$

Signs in each quadrant

3. State the quadrant or quadrants in which each condition holds.

(a) $\sin\theta > 0$ (b) $\cos\theta < 0$
(c) $\tan\theta > 0$ (d) $\sin\theta < 0$ and $\cos\theta > 0$

The unit circle and gradients

4. The point P on the unit circle has angle θ.

(a) State the coordinates of P in terms of θ.
(b) Explain why $\cos^2\theta + \sin^2\theta = 1$ for every θ.
(c) Explain why $\tan\theta$ is undefined at $\theta = \frac{\pi}{2}$.

5. Lines through the origin, using $y = x\tan\theta$. Give each angle in radians.

(a) Write the equation of the line through the origin making an angle of $\frac{\pi}{3}$ with the positive x-axis.
(b) Find the angle a line of gradient 1 makes with the positive x-axis.
(c) Find the anticlockwise angle θ from the positive x-axis to the line $y = -x$, with $0 \le \theta < \pi$.

Concept check

6. A student writes $\cos\frac{2\pi}{3} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$, taking the reference angle from the nearest axis. Identify the error and find the correct value.

8.2 Trigonometric Identities

An **identity** is an equation true for every angle. The first one is Pythagoras' theorem read off the unit circle: the point $(\cos\theta, \sin\theta)$ is distance 1 from the origin, so $\cos^2\theta + \sin^2\theta = 1$.

KEY · IB $\cos^2\theta + \sin^2\theta = 1.$

8.2.1 Double and compound angles

The functions of a sum of two angles can be expanded, and the double-angle formulas follow from the sum formulas. These **compound-angle** identities are usually just quoted, but the proof is short and uses only the unit circle and the cosine rule from Ch. 7.

Put two points on the unit circle, P at angle A and Q at angle B. Their coordinates are $P(\cos A,\ \sin A)$ and $Q(\cos B,\ \sin B)$, and the angle $\angle POQ$ at the centre is $A - B$.

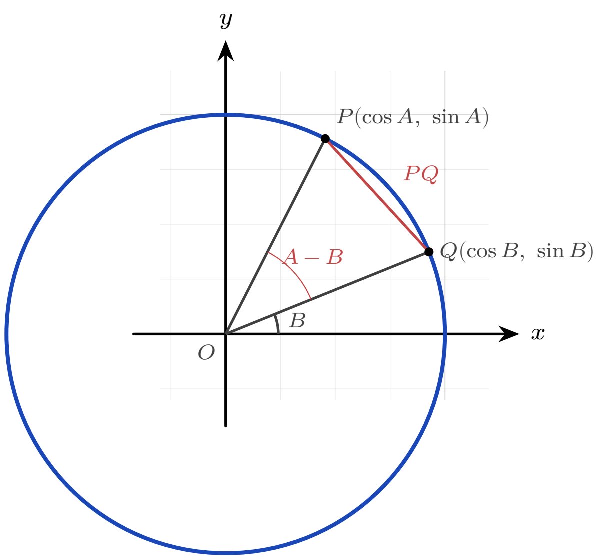

Figure 8.3 Two points on the unit circle, at angles A and B. The angle between the radii is $A - B$, and the chord PQ can be measured either by the distance formula or by the cosine rule.

Now find the distance PQ in two different ways.

By the distance formula:

$$PQ^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2.$$

Expanding gives $\cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B$. Both Pythagorean pairs collapse to 1, so

$$PQ^2 = 2 - 2(\cos A\cos B + \sin A\sin B).$$

By the cosine rule in triangle OPQ, where both sides from the centre have length 1 and the angle between them is $A-B$:

$$PQ^2 = 1^2 + 1^2 - 2(1)(1)\cos(A-B) = 2 - 2\cos(A-B).$$

The two expressions are the same length, so

$$\cos(A-B) = \cos A\cos B + \sin A\sin B.$$

Every other compound-angle formula follows from this one. Replacing B by $-B$, and using $\cos(-B) = \cos B$ with $\sin(-B) = -\sin B$, gives the formula for $\cos(A+B)$.

A sine is a cosine a quarter turn away: $\sin\theta = \cos(\frac{\pi}{2} - \theta)$. That single fact is all the sine version needs. Write the sum in that form, then group the result as a *difference* of two angles:

$$\sin(A+B) = \cos(\tfrac{\pi}{2} - A - B) = \cos\big((\tfrac{\pi}{2} - A) - B\big).$$

That is exactly the shape just proved. Expanding it, and using $\cos(\frac{\pi}{2} - A) = \sin A$ together with $\sin(\frac{\pi}{2} - A) = \cos A$, gives

$$\sin(A+B) = \sin A\cos B + \cos A\sin B.$$

The tangent versions come from dividing the sine formula by the cosine formula.

KEY · IB HL **Compound angles:**

$$\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B, \qquad \cos(A\pm B) = \cos A\cos B \mp \sin A\sin B,$$

$$\tan(A\pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}.$$

Setting $B = A$ reduces each of these to a **double-angle** identity. Working through the derivation once shows there is nothing new to learn here.

Put $B = A$ in the sine formula:

$$\sin 2A = \sin(A+A) = \sin A\cos A + \cos A\sin A = 2\sin A\cos A.$$

Do the same in the cosine formula:

$$\cos 2A = \cos(A+A) = \cos A\cos A - \sin A\sin A = \cos^2 A - \sin^2 A.$$

That is the first form. The other two come from $\sin^2 A + \cos^2 A = 1$, used to remove one function or the other. Replacing $\sin^2 A$ by $1 - \cos^2 A$ gives

$$\cos 2A = \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1,$$

and replacing $\cos^2 A$ by $1 - \sin^2 A$ instead gives

$$\cos 2A = (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A.$$

The tangent version follows in the same way, by putting $B = A$ in the tangent formula.

The sine and cosine versions sit in the SL half of the booklet; the tangent version is HL.

KEY · IB **Double angles:**

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$$

KEY · IB HL $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}.$

The three forms of $\cos 2\theta$ are one identity rewritten by $\sin^2 + \cos^2 = 1$. Choose whichever leaves you with only sines or only cosines. That choice is what makes the identity so useful when solving equations.

The compound-angle formulas also give exact values beyond the basic table. Write the angle as a sum or difference of two special angles, such as $165^\circ = 120^\circ + 45^\circ$, and expand. The two angles need not be acute; carry the sign of each ratio from its quadrant.

Worked through. To find $\sin 165^\circ$ exactly, write $165^\circ = 120^\circ + 45^\circ$ and use the compound-angle formula:

$$\sin 165^\circ = \sin 120^\circ \cos 45^\circ + \cos 120^\circ \sin 45^\circ = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \left(-\frac{1}{2}\right) \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}.$$

Here $\cos 120^\circ = -\frac{1}{2}$ carries a minus sign from the second quadrant, and that minus sign is what turns the sum into a difference.

Identities also let you move between ratios *without ever finding the angle*. The angle stays unknown from start to finish, and the Pythagorean identity supplies everything needed. The identity gives the square of the unknown ratio, so it leaves a choice of sign. Let the quadrant of the angle fix that sign before you use the value.

EXAMPLE 8.1

Given that $\cos x = \dfrac{3}{4}$ and x is acute, find the exact value of $\sin 2x$ without finding x.

From $\cos^2 x + \sin^2 x = 1$, $\sin^2 x = 1 - \frac{9}{16} = \frac{7}{16}$, and since x is acute, $\sin x = \frac{\sqrt{7}}{4}$ (positive). Then

$$\sin 2x = 2\sin x\cos x = 2 \cdot \frac{\sqrt{7}}{4} \cdot \frac{3}{4} = \frac{3\sqrt{7}}{8}.$$

The word "acute" fixed the sign of $\sin x$. If x had instead been in the fourth quadrant, the same working would give $\sin 2x = -\frac{3\sqrt{7}}{8}$: always let the quadrant choose the sign before doubling.

Proving an identity uses the same substitutions: start from one side, substitute known identities, and simplify until the other side appears.

EXAMPLE 8.2

Prove that $\cos(A-B) - \cos(A+B) = 2\sin A \sin B$.
Work on the left side, expanding each cosine by the compound-angle formula:

$$\cos(A-B) = \cos A\cos B + \sin A \sin B, \qquad \cos(A+B) = \cos A \cos B - \sin A \sin B.$$

The two expansions differ only in the sign of the second term, so subtracting removes the $\cos A \cos B$ pair and doubles what is left:

$$\cos(A-B) - \cos(A+B) = 2\sin A \sin B. \qquad \blacksquare$$

Expanding the more complicated side first is the usual choice. Once both expansions are written down, the terms that have to cancel are visible before any algebra is done.

8.2.2 One wave from two

This subsection is beyond the syllabus. It is included because the method is short and it appears in physics.

A sum such as $8\sin x + 15\cos x$ mixes a sine and a cosine of the same angle. Both have period 2π, so the sum repeats every 2π as well, and it turns out to be a single sine wave, stretched and shifted. Reading the compound-angle formula backwards shows why.

Expanding $R\sin(x+\alpha)$ gives

$$R\sin(x+\alpha) = R\cos\alpha\,\sin x + R\sin\alpha\,\cos x,$$

which already has the shape $a\sin x + b\cos x$, with $a = R\cos\alpha$ and $b = R\sin\alpha$. Squaring those two and adding removes α, since $\cos^2\alpha + \sin^2\alpha = 1$, and leaves $a^2 + b^2 = R^2$. Dividing them removes R instead and leaves $\tan\alpha = \frac{b}{a}$. So any such sum can be written as one wave, and the two constants can be recovered from a and b alone.

KEY **The wave form.** For constants a and b not both zero,

$$a\sin x + b\cos x = R\sin(x+\alpha), \qquad R = \sqrt{a^2+b^2}, \qquad \tan\alpha = \frac{b}{a},$$

with $R > 0$ and α taken in the quadrant where $\cos\alpha$ has the sign of a and $\sin\alpha$ the sign of b. The same R and α give the cosine form

$$a\cos x + b\sin x = R\cos(x-\alpha).$$

Since $\sin$ and $\cos$ run between -1 and 1, the expression has maximum R and minimum $-R$.

This form is not printed in the formula booklet, so rebuild it from the expansion of $R\sin(x+\alpha)$ whenever it is needed. Its value is that the variable now appears once instead of twice, which turns a maximum, a minimum or an equation into the ordinary work of §8.6.

Worked through. To write $8\sin x + 15\cos x$ in the form $R\sin(x+\alpha)$ with $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, state its maximum and minimum values, and solve $8\sin x + 15\cos x = 6$ for $0 \le x \le 2\pi$, take $a = 8$ and $b = 15$. Both are positive, so α is in the first quadrant:

$$R = \sqrt{8^2+15^2} = \sqrt{289} = 17, \qquad \tan\alpha = \frac{15}{8} \implies \alpha = \arctan\frac{15}{8} = 1.0808,$$

giving $8\sin x + 15\cos x = 17\sin(x+1.0808)$.

The sine factor runs between -1 and 1, so the maximum is 17 and the minimum is -17.

For the equation, $17\sin(x+\alpha) = 6$, so $\sin(x+\alpha) = \frac{6}{17}$. Put $u = x+\alpha$. As x runs over $[0,\ 2\pi]$, u runs over $[1.0808,\ 7.3640]$. The reference angle is $\arcsin\frac{6}{17} = 0.3607$, and $\frac{6}{17}$ is positive, so sine takes it in the first and second quadrants:

$$u = 0.3607 \qquad \text{and} \qquad u = \pi - 0.3607 = 2.7809.$$

The first sits below 1.0808, so add a turn: $0.3607 + 2\pi = 6.6439$, which is inside. The second is already inside, and a further turn from it would pass 7.3640. So $u = 2.7809$ or $u = 6.6439$, and subtracting α,

$$x = 1.70 \qquad \text{or} \qquad x = 5.56 \quad (3\text{ s.f.}).$$

Transforming the interval along with the substitution is what saved the second answer. Had u been left to run over $[0,\ 2\pi]$, the pair 0.3607 and 2.7809 would have given one negative x and lost 5.56 altogether.

YOUR TURN 8.2 Trigonometric Identities

Recognising the identities

7. Simplify each expression.

(a) $\sin^2\theta + \cos^2\theta$
(b) $2\sin\theta\cos\theta$
(c) $\cos^2\theta - \sin^2\theta$
(d) $1 - 2\sin^2\theta$

Values without the angle

8. Given $\sin\theta = \dfrac{3}{5}$ with θ acute, find each exact value.

(a) $\cos\theta$
(b) $\tan\theta$
(c) $\sin 2\theta$
(d) $\cos 2\theta$

9. Given $\cos\theta = -\dfrac{3}{5}$ with θ in the second quadrant, find each exact value.

(a) $\sin\theta$
(b) $\tan\theta$
(c) $\sin 2\theta$
(d) $\cos 2\theta$

Compound-angle values

10. Use a compound-angle formula to find each exact value.

(a) $\cos 15^\circ$, from $45^\circ - 30^\circ$
(b) $\cos 105^\circ$, from $60^\circ + 45^\circ$
(c) $\cos 165^\circ$, from $120^\circ + 45^\circ$

Tangent formulas

11. Use the tangent compound-angle and double-angle formulas.

(a) Find the exact value of $\tan 75^\circ$, from $45^\circ + 30^\circ$.
(b) Given $\tan A = \dfrac{1}{2}$ and $\tan B = \dfrac{1}{3}$, with A and B acute, find $\tan(A + B)$, and hence $A + B$.
(c) Given $\tan\theta = \dfrac{3}{4}$, find the exact value of $\tan 2\theta$.
(d) Given $\tan\theta = -2$, find the exact value of $\tan 2\theta$.

Proving identities

12. Prove each identity.

(a) $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$

(b) $\dfrac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$

(c) $(\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$

(d) $\dfrac{\cos 2\theta}{\cos\theta + \sin\theta} = \cos\theta - \sin\theta$

(e) $\tan\theta \sin 2\theta = 2\sin^2\theta$

(f) $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2$

Concept check

13. A student writes $\sin(A+B) = \sin A + \sin B$.

(a) Give a counter-example.

(b) State the correct expansion of $\sin(A+B)$.

8.3 Graphs of Trigonometric Functions

Each graph plots one coordinate of P against the angle. Send the point P round the circle and plot its y-coordinate against the angle: that plot *is* $y = \sin x$. Plot its x-coordinate instead and you get $y = \cos x$.

Every feature of the sine graph can be read off the circle before you plot a single point.

- The y-coordinate of P never leaves -1 to 1, so the curve stays between those two heights. Its **amplitude**, the distance from the middle to the top, is 1.
- P returns to its starting place after one full turn, so the curve repeats every 2π. Its **period**, the width of one complete cycle, is 2π.
- $y = 0$ when P sits on the x-axis, at $\theta = 0$, π and 2π, so the curve crosses the axis at those three angles.
- y is largest at the top of the circle, at $\theta = \frac{\pi}{2}$, so the maximum is 1 and it occurs at $x = \frac{\pi}{2}$.
- $y > 0$ everywhere in the upper half of the circle, so the curve is above the axis for $0 < x < \pi$.

Read the x-coordinate instead and the cosine graph appears the same way. It starts at 1, because P starts on the positive x-axis, and falls to 0 at $\frac{\pi}{2}$, where P is at the top and its x-coordinate is zero.

So both curves have amplitude 1 and period 2π. They differ only in where a cycle begins.

The x-coordinate at angle θ is the y-coordinate a quarter turn later, so $\cos\theta = \sin(\theta + \frac{\pi}{2})$, and the cosine curve is the sine curve shifted left by $\frac{\pi}{2}$.

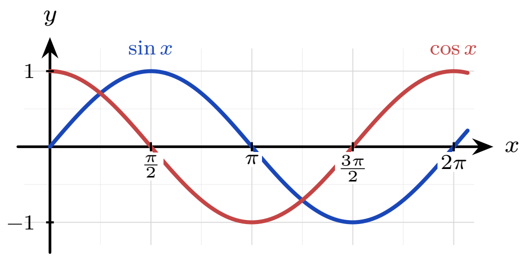

Figure 8.4 One turn of the circle unrolled. Both waves have amplitude 1 and period 2π; the cosine is the sine shifted left by $\frac{\pi}{2}$.

The tangent behaves differently, and the circle explains why. Since $\tan\theta = \frac{y}{x}$, it is undefined wherever $x = 0$, which is at the top and the bottom of the circle. That gives vertical asymptotes at $x = \frac{\pi}{2} + n\pi$.

Its period is π, not 2π. Half a turn sends P to the point diametrically opposite, whose coordinates are $(-x, -y)$. The ratio $\frac{-y}{-x}$ equals $\frac{y}{x}$, so the tangent repeats after only half a revolution.

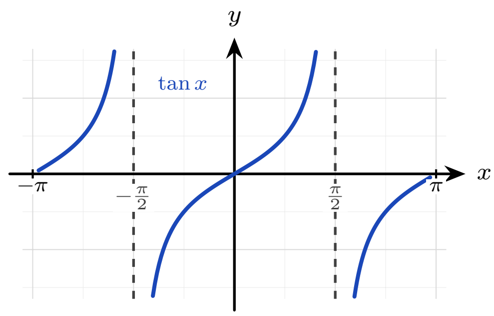

Figure 8.5 The tangent repeats every π, not every 2π, with a vertical asymptote wherever $\cos x = 0$.

That is the basic behaviour, collected in one place.

	$\sin x$	$\cos x$	$\tan x$
Domain	all x	all x	$x \neq \frac{\pi}{2} + n\pi$
Range	$-1 \leq y \leq 1$	$-1 \leq y \leq 1$	all y
Period	2π	2π	π
Amplitude	1	1	none
Zeros	$n\pi$	$\frac{\pi}{2} + n\pi$	$n\pi$

Here n stands for any integer, and the last row gives the values of x at which the curve crosses the axis. The tangent has no amplitude because it has no largest or smallest value. Amplitude measures half the rise and fall, and the tangent rises without limit.

8.3.1 Transforming the wave

The general sinusoid stretches and slides the basic sine, and each parameter has a plain meaning.

KEY For $y = a\sin(b(x+c)) + d$:

$$\text{amplitude} = |a|, \qquad \text{period} = \frac{2\pi}{|b|},$$

$$\text{horizontal shift} = -c, \qquad \text{vertical shift (midline)} = d.$$

A question can also run the other way, giving you the graph's largest and smallest values and asking for a and d. Since the maximum is $d + |a|$ and the minimum is $d - |a|$, adding and subtracting recovers both constants.

- **Amplitude.** From the equation it is $|a|$. From a graph it is $\dfrac{\text{max} - \text{min}}{2}$, half the total rise and fall.
- **Midline.** From the equation it is the line $y = d$. From a graph it is $y = \dfrac{\text{max} + \text{min}}{2}$, the level halfway between the top and the bottom.
- **Maximum and minimum.** They are $d + |a|$ and $d - |a|$: the midline, plus and minus the amplitude.
- **Period.** From the equation it is $\dfrac{2\pi}{|b|}$. From a graph it is the horizontal distance from one repeat to the next.

Read the first two as sentences and they stop being formulas to memorise. Consider the wave $y = 5\sin(2x) - 3$, whose maximum is 2 and whose minimum is -8. The amplitude is $\frac{2-(-8)}{2} = 5$ and the midline is $\frac{2+(-8)}{2} = -3$, which is what the equation says.

Worked through. In $y = 2\sin(3x) + 1$ the amplitude is 2, so the wave rises and falls 2 either side of its midline. The period is $\frac{2\pi}{3}$, so it completes a full cycle three times as fast as $\sin x$. The midline is $y = 1$. The graph therefore runs from $1 - 2 = -1$ up to $1 + 2 = 3$, giving range $-1 \le y \le 3$.

8.3.2 Symmetry of the graphs HL

Each reflection symmetry of the circle from Section 8.1 becomes a visible symmetry of the graphs. The cosine curve is mirror-symmetric in the y-axis, so $\cos(-x) = \cos x$: cosine is an **even** function. The sine and tangent curves map onto themselves under a half-turn about the origin, so $\sin(-x) = -\sin x$ and $\tan(-x) = -\tan x$: both are **odd**, in the sense of Ch. 5. The same circle symmetries give the supplementary-angle relations.

KEY HL $\sin(\pi - \theta) = \sin\theta, \qquad \cos(\pi - \theta) = -\cos\theta, \qquad \tan(\pi - \theta) = -\tan\theta.$

You can see all three on the graphs directly: the sine curve is symmetric about the vertical line $x = \frac{\pi}{2}$, while cosine and tangent flip sign across it. In an exam these relations remove the need for a calculator: $\cos\frac{3\pi}{4} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$.

YOUR TURN 8.3 Graphs of Trigonometric Functions

Amplitude, period and midline

14. State the amplitude, the period and the midline of each graph.

(a) $y = 3\sin x$

(b) $y = \sin 2x$

(c) $y = 4\sin(3x) - 2$

(d) $y = 5\sin(4x) + 3$

15. State the range of each function.

(a) $y = 2\sin x + 1$

(b) $y = 3\sin(2x) - 1$

(c) $y = 2\cos\left(\frac{x}{2}\right) - 1$

Horizontal shift

16. State the amplitude, the period, the horizontal shift and the midline of each graph. *(Hint: write the argument in the form $b(x+c)$ first)*

(a) $y = 3\sin\left(x - \frac{\pi}{4}\right) + 1$

(b) $y = 2\sin\left(2x + \frac{\pi}{3}\right)$

(c) $y = 5\sin(3x - \pi) - 2$

From the maximum and minimum

17. Find the constants from the information given.

(a) The graph of $y = a\sin(bx) + d$, with $a > 0$ and $b > 0$, has maximum value 7, minimum value -1 and period π. Find a, b and d.

(b) The depth of water in a harbour is modelled by $h(t) = a\sin(bt) + d$ metres, t hours after midnight, with $a > 0$ and $b > 0$. The greatest depth is 12.5 m, the least depth is 2.5 m, and the pattern repeats every 12 hours. Find a, b and d.

The tangent graph

18. Answer each question about $y = \tan x$.

(a) State its period.

(b) State the equations of two of its asymptotes.

(c) Explain why the asymptotes occur where they do.

Symmetry of the graphs

19. An angle θ has $\sin\theta = 0.6$ and $\cos\theta = 0.8$. Use the symmetries of the graphs to write down each value.

(a) $\sin(\pi - \theta)$ (b) $\cos(\pi - \theta)$
(c) $\tan(\pi - \theta)$ (d) $\sin(-\theta)$

Modelling

20. A wheel of diameter 40 m has its lowest point 2 m above the ground and turns once every 20 seconds. A seat starts at the bottom. Its height in metres after t seconds is modelled by $h(t) = a\sin(b(t+c)) + d$, with $a > 0$ and $b > 0$.

(a) State the amplitude a and the midline value d.
(b) Find the value of b.
(c) The seat is at its lowest point when $t = 0$. Show that $c = -5$ is a suitable value, and write down $h(t)$.
(d) Find the height after 8 seconds.

Concept check

21. A student says: "*the graph of* $y = \sin\left(2x - \dfrac{\pi}{3}\right)$ *is the graph of* $y = \sin 2x$ *translated* $\dfrac{\pi}{3}$ *to the right.*" Identify the error and state the correct translation.

22. A sinusoidal graph has maximum value 7 and minimum value -1. A student says: "*the amplitude is* 8." Identify the error, and find the correct amplitude and the equation of the midline.

8.4 Reciprocal Trigonometric Functions HL

Divide 1 by a sine, a cosine or a tangent and three more functions appear. Each has a name of its own, a graph with vertical asymptotes, and an identity that makes certain equations solvable.

KEY · IB $\sec\theta = \dfrac{1}{\cos\theta}, \qquad \operatorname{cosec}\theta = \dfrac{1}{\sin\theta}.$

The third is the **cotangent**, $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$. Its definition is not printed in the formula booklet, so learn it.

The pairing looks wrong at first: *sec*ant goes with *co*sine, and *cosec*ant with *sin*e. The *co* means complement: $\operatorname{cosec}\theta = \sec(\frac{\pi}{2} - \theta)$, just as $\cos\theta = \sin(\frac{\pi}{2} - \theta)$. As a quick check, use the third letter. In **se<u>c</u>** the third letter is c, for cosine. In **cose<u>c</u>** it is s, for sine. In **co<u>t</u>** it is t, for tangent.

Two facts follow from the reciprocal alone, and between them they fix every graph.

First, a reciprocal has an asymptote wherever the original function is zero, because you cannot divide by zero. So $\sec$ has asymptotes where $\cos\theta = 0$, at $\theta = \frac{\pi}{2} + n\pi$, while cosec and $\cot$ have them where $\sin\theta = 0$, at $\theta = n\pi$.

Second, $\sin$ and $\cos$ never exceed 1 in size, so their reciprocals are never smaller than 1 in size. Neither $\sec\theta$ nor $\operatorname{cosec}\theta$ ever takes a value strictly between -1 and 1. State that gap whenever you give the range.

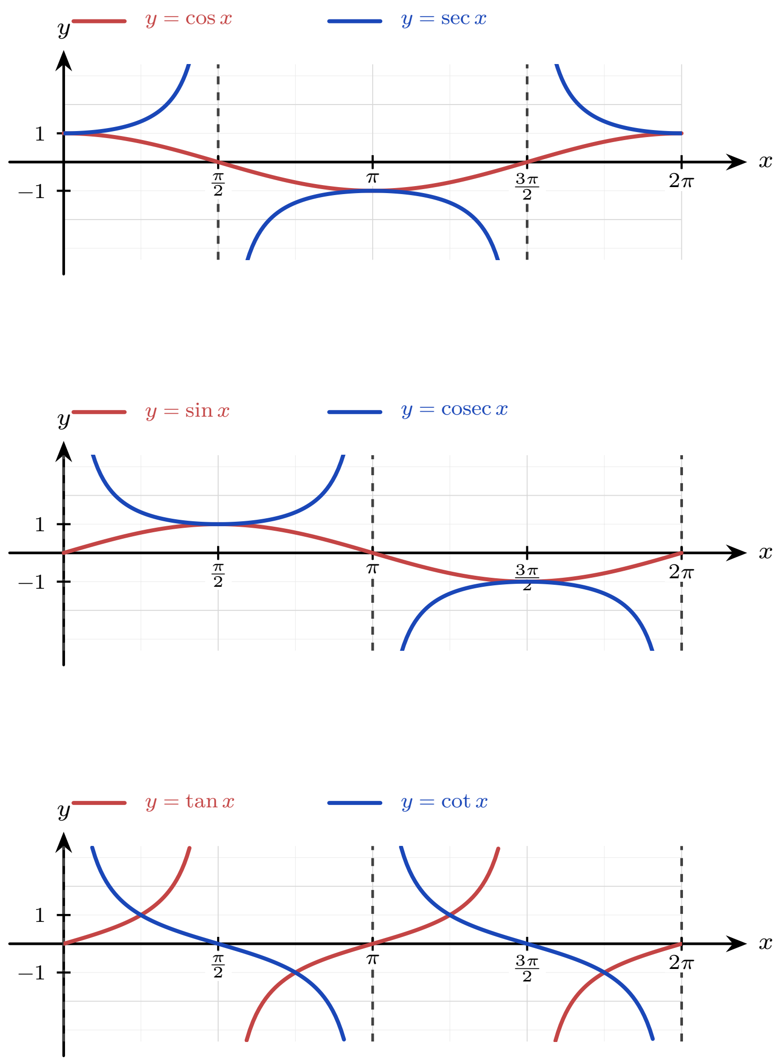

Figure 8.6 In each panel the parent function is scarlet and its reciprocal cobalt. The reciprocal has an asymptote, marked by a dashed line, wherever its parent crosses the axis, and the two curves meet wherever the parent reaches ± 1. Secant and cosecant never enter the band between -1 and 1; cotangent has no such gap.

Taking n to be any integer, the three read as follows.

Function	Domain	Range	Period
$\sec\theta = \dfrac{1}{\cos\theta}$	all θ except $\frac{\pi}{2} + n\pi$	$y \le -1$ or $y \ge 1$	2π
$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$	all θ except $n\pi$	$y \le -1$ or $y \ge 1$	2π
$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	all θ except $n\pi$	all real y	π

Every excluded value in the domain column is a zero of the function underneath, and every one of them is an asymptote on the graph.

8.4.1 The two Pythagorean variants

The identity $\cos^2\theta + \sin^2\theta = 1$ has two companions, one for each reciprocal pair, and both come from dividing it through.

Dividing $\cos^2\theta + \sin^2\theta = 1$ by $\cos^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta,$$

since $\frac{\sin\theta}{\cos\theta} = \tan\theta$ and $\frac{1}{\cos\theta} = \sec\theta$. Dividing the same identity by $\sin^2\theta$ instead gives the companion result. Both are in the formula booklet; the derivation shows where they come from.

KEY · IB $1 + \tan^2\theta = \sec^2\theta, \qquad 1 + \cot^2\theta = \operatorname{cosec}^2\theta.$

These identities exist to turn mixed equations into single-function ones, exactly as the double angles did.

Worked through. The equation $\tan^2 x + \sec x = 1$, for $0 \le x \le 2\pi$, mixes $\tan$ and $\sec$; the identity $\tan^2 x = \sec^2 x - 1$ rewrites it in $\sec x$ alone:

$$\sec^2 x - 1 + \sec x = 1 \implies \sec^2 x + \sec x - 2 = 0 \implies (\sec x + 2)(\sec x - 1) = 0.$$

So $\sec x = -2$ or $\sec x = 1$, which is $\cos x = -\frac{1}{2}$ or $\cos x = 1$. From the circle,

$$x = \frac{2\pi}{3},\ \frac{4\pi}{3} \quad \text{or} \quad x = 0,\ 2\pi.$$

Converting back to $\cos$ at the end is a sensible final step, because the unit circle is read in sine and cosine, not in secant.

YOUR TURN 8.4 Reciprocal Trigonometric Functions HL

Exact values

23. Evaluate each exactly.

(a) $\sec\frac{\pi}{3}$

(b) $\operatorname{cosec}\frac{\pi}{2}$

(c) $\cot\frac{\pi}{4}$

(d) $\sec\frac{2\pi}{3}$

(e) $\cot\frac{2\pi}{3}$

(f) $\operatorname{cosec}\frac{7\pi}{6}$

Graphs of the reciprocal functions

24. Use the graphs of the reciprocal functions.

(a) State the range of $y = \sec x$.

(b) State the equations of the asymptotes of $y = \operatorname{cosec} x$ for $0 \le x \le 2\pi$.

(c) State the domain and the range of $y = \cot x$.

(d) Find the equations of the asymptotes of $y = \sec 2x$ for $0 \le x \le \pi$.

Simplifying

25. Simplify each expression.

(a) $\sec^2\theta - \tan^2\theta$

(b) $\cot\theta\sin\theta$

(c) $\dfrac{1}{\cot\theta}$

(d) $(\sec\theta - 1)(\sec\theta + 1)$

(e) $\dfrac{\sec\theta}{\operatorname{cosec}\theta}$

(f) $\sin\theta\operatorname{cosec}\theta + \cos\theta\sec\theta$

Proving identities

26. Prove each identity.

(a) $\cot\theta\,\sec\theta = \operatorname{cosec}\theta$

(b) $\dfrac{\operatorname{cosec}\theta}{\cot\theta + \tan\theta} = \cos\theta$

(c) $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta\operatorname{cosec}^2\theta$

(d) $\dfrac{1+\cot^2\theta}{1+\tan^2\theta} = \cot^2\theta$

Solving equations

27. Solve for $0 \le x \le 2\pi$.

(a) $\sec x = 2$

(b) $2\tan^2 x + \sec^2 x = 4$

(c) $\cot^2 x + \operatorname{cosec} x = 1$

Concept check

28. A student writes: "$\cos^{-1} 0 = \dfrac{1}{\cos 0} = 1$." Explain the difference between $\cos^{-1} x$ and $\sec x$, and evaluate both at $x = 0$.

8.5 Inverse Trigonometric Functions HL

Everything so far has gone one way: give an angle, get a ratio. The inverse functions run the other way, recovering the angle from the ratio.

You have already used them without naming them. When you solve $\sin x = \frac{1}{2}$ and press the $\sin^{-1}$ key, that key is the inverse function.

Two notations, one function. The inverse of sine is written either $\arcsin x$ or $\sin^{-1} x$. They mean exactly the same thing. These notes use $\arcsin$, because the other form invites a serious confusion:

$$\sin^{-1} x \text{ means the } \textit{inverse}, \qquad \text{but} \qquad (\sin x)^{-1} = \frac{1}{\sin x} = \operatorname{cosec} x.$$

The -1 is not an exponent here. Writing $\arcsin$, $\arccos$ and $\arctan$ removes the ambiguity, and your calculator's $\sin^{-1}$ key is the same function under the other name.

8.5.1 Why a restriction is needed

There is an immediate difficulty. A function has an inverse only if it is one-to-one (Ch. 5), and $\sin$, $\cos$ and $\tan$ are not. Each takes every value in its range infinitely often, so the horizontal line $y = \frac{1}{2}$ meets $y = \sin x$ infinitely many times.

That is a real problem, not a technicality. If you ask "which angle has sine $\frac{1}{2}$?", there is no single answer: $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{13\pi}{6}$ and infinitely many others all qualify. A function must return one output, so it has to choose.

The fix is the one from Ch. 5: cut the domain down until the function is one-to-one, then invert what is left. Which piece to keep is a choice, and mathematicians have agreed on one for each function. Those agreed pieces are shaded in the diagrams below, and they are what fix the ranges in the table.

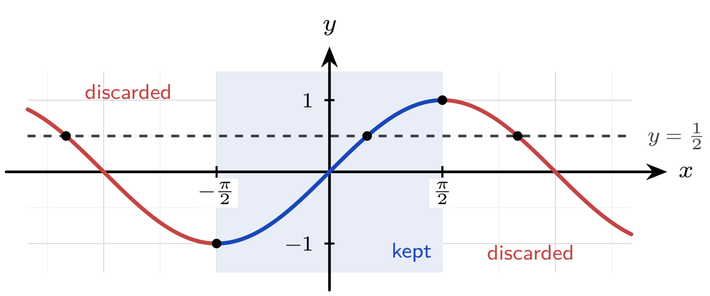

Figure 8.7 The dashed line $y = \frac{1}{2}$ meets $y = \sin x$ again and again, which is why "the angle whose sine is $\frac{1}{2}$" has no single answer. Keeping only the shaded piece from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ leaves one intersection. On that piece the curve rises steadily, so it is one-to-one and every output comes from exactly one input.

The same is done for the other two: cosine keeps $0 \le x \le \pi$, where it falls steadily, and tangent keeps $-\frac{\pi}{2} < x < \frac{\pi}{2}$, one branch between two asymptotes. Those kept pieces become the *ranges* of the inverses, because domain and range swap when a function is inverted.

Function	Domain	Range
$\arcsin x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$\arccos x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$\arctan x$	all real x	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

Each inverse is a reflection of its restricted parent in the line $y = x$ (Ch. 5), so the graphs are the familiar curves turned on their sides. Arcsine climbs steeply from $(-1, -\frac{\pi}{2})$ to $(1, \frac{\pi}{2})$. Arccosine falls from $(-1, \pi)$ to $(1, 0)$. Arctangent flattens towards the horizontal asymptotes $y = \pm\frac{\pi}{2}$, which are the vertical asymptotes of tangent after the reflection.

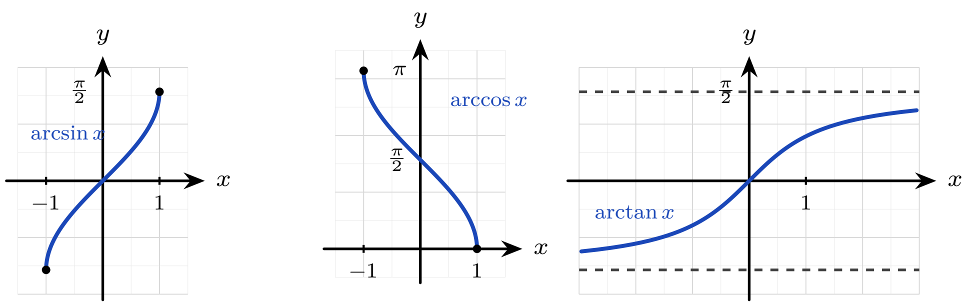

Figure 8.8 Each inverse is its restricted parent reflected in $y = x$. The marked endpoints show the range each one is confined to.

Worked through. To find $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$, look for the angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine is $-\frac{\sqrt{3}}{2}$: it is $-\frac{\pi}{3}$. Not $\frac{4\pi}{3}$ and not $\frac{5\pi}{3}$: those solve the equation but lie outside arcsine's range. In the same way, $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ is the angle in $[0, \pi]$ whose cosine is $-\frac{\sqrt{3}}{2}$, which is $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$. Arccosine handles negatives by reaching into the second quadrant. Finally, $\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$, and $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$, so $\arcsin(\sin\frac{5\pi}{4})$ returns $-\frac{\pi}{4}$, not $\frac{5\pi}{4}$: arcsine can only answer with angles from its own range.

CAUTION $\arcsin(\sin x)$ does not always give back x. Every answer from arcsin must lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and 2 does not. So $\arcsin(\sin 2) = \pi - 2$, an angle with the same sine as 2 but inside the range. The restriction is working, not failing.

8.5.2 Compositions the other way round

A composition can also run with the inverse on the inside, as in $\cos(\arcsin k)$. The angle is never found here, and it never needs to be. Name it: let $\theta = \arcsin k$, so $\sin\theta = k$. A right triangle with opposite k and hypotenuse 1 has exactly that sine, and Pythagoras gives its adjacent side as $\sqrt{1-k^2}$, so

$$\cos(\arcsin k) = \sqrt{1-k^2}.$$

The positive root is the right one because $\arcsin$ answers in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and cosine is never negative there. The same triangle serves $\arccos$ and $\arctan$: place the known ratio on the two sides it names, find the third by Pythagoras, then read off whatever the question asks for.

EXAMPLE 8.3

Find the exact value of (a) $\cos\left(\arcsin \dfrac{20}{29}\right)$, (b) $\sin\left(\arccos \dfrac{9}{41} + \arctan \dfrac{20}{21}\right)$.

(a) Let $\theta = \arcsin \frac{20}{29}$, so $\sin\theta = \frac{20}{29}$: opposite 20, hypotenuse 29. The adjacent side is $\sqrt{29^2 - 20^2} = \sqrt{441} = 21$, so

$$\cos\left(\arcsin \frac{20}{29}\right) = \frac{21}{29},$$

positive because θ lies in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(b) Name both angles: $A = \arccos \frac{9}{41}$ and $B = \arctan \frac{20}{21}$. A 9–40–41 triangle gives $\cos A = \frac{9}{41}$ and $\sin A = \frac{40}{41}$, taken positive because $\arccos$ answers in $[0, \pi]$, where sine is never negative. A 20–21–29 triangle, the same one as in part (a), gives $\sin B = \frac{20}{29}$ and $\cos B = \frac{21}{29}$, both positive because $\tan B > 0$ puts B in the first quadrant. The sum is now a compound angle:

$$\sin(A+B) = \sin A\cos B + \cos A \sin B = \frac{40}{41}\cdot\frac{21}{29} + \frac{9}{41}\cdot\frac{20}{29} = \frac{840+180}{1189} = \frac{1020}{1189}.$$

Neither A nor B was ever evaluated. The range of each inverse function is what fixes the sign of every side, and that is the only place in the working where a decision is made.

YOUR TURN 8.5 Inverse Trigonometric Functions HL

Exact values

29. Evaluate each exactly.

(a) $\arcsin \dfrac{1}{2}$ (b) $\arccos 0$

(c) $\arctan(-1)$ (d) $\arccos\left(-\dfrac{1}{2}\right)$

Domains, ranges and graphs

30. Use the graphs of the inverse functions.

(a) Sketch the graph of $y = \arccos x$, and label the coordinates of its endpoints.
(b) Write down the equations of the horizontal asymptotes of $y = \arctan x$.
(c) Find the domain of $y = \arcsin(x-2)$.
(d) Find the range of $y = 2\arctan x + \pi$.

The inverse on the outside

31. An inverse undoes its function only on the piece that was kept.

(a) State the range of $\arcsin x$, of $\arccos x$ and of $\arctan x$.

(b) Evaluate $\arcsin\left(\sin\dfrac{3\pi}{4}\right)$, and explain why the answer is not $\dfrac{3\pi}{4}$.

(c) Evaluate $\arccos\left(\cos\dfrac{4\pi}{3}\right)$.

(d) Evaluate $\arctan\left(\tan\dfrac{5\pi}{6}\right)$.

(e) State every value of x for which $\arcsin(\sin x) = x$.

32. Find the exact value, or write **undefined**.

(a) $\arcsin\left(\sin\dfrac{2\pi}{3}\right)$

(b) $\arccos(\cos 4)$

(c) $\arctan\left(\tan\left(-\dfrac{3\pi}{4}\right)\right)$

(d) $\arcsin(\sin 3)$

(e) $\arcsin\left(\tan\dfrac{\pi}{3}\right)$

(f) $\arctan\left(2\sin\dfrac{\pi}{3}\right)$

The inverse on the inside

33. Find the exact value, or write **undefined**.

(a) $\tan(\arctan 12)$

(b) $\cos\left(\arccos\dfrac{2\pi}{3}\right)$

(c) $\sin(\arcsin \pi)$

(d) $\sin\left(\arctan\dfrac{3}{4}\right)$

(e) $\cos\left(\arcsin\dfrac{7}{25}\right)$

(f) $\cos\left(\arctan\dfrac{1}{2}\right)$

(g) $\cos(\arcsin 0.6)$

Sums of inverse angles

34. Use a compound-angle formula to find each exact value.

(a) $\sin\left(\arccos\dfrac{3}{5} + \arctan\dfrac{5}{12}\right)$

(b) $\cos\left(\arctan 3 + \arcsin\dfrac{1}{3}\right)$

Concept check

35. A student says: "$\arcsin\left(\sin\dfrac{5\pi}{6}\right) = \dfrac{5\pi}{6}$, *because* $\arcsin$ *undoes* $\sin$." Explain why this is false, and find the correct value.

8.6 Solving Trigonometric Equations

An equation like $\sin x = \frac{1}{2}$ has not one solution but infinitely many, because the sine repeats every 2π. Your calculator returns only one of them, the **principal value**. Everything else you must find yourself, and the circle is what tells you how.

Start by counting the solutions in a single turn. Solving $\sin x = k$ means finding the points on the circle whose y-coordinate is k, so draw the horizontal line at height k and see where it meets the circle. Three things can happen.

- If $-1 < k < 1$ the line cuts the circle in **two** places, so two angles in each turn have sine k.
- If $k = 1$ or $k = -1$ the line touches the circle at **one** point, the top or the bottom. The two angles have merged into one.
- If $|k| > 1$ the line misses the circle, and there is **no** solution at all.

The same three cases hold for $\cos x = k$, read with a vertical line at $x = k$ instead. Tangent behaves differently: it takes every real value, so $\tan x = k$ has solutions whatever k is.

Two words are needed next. You will use both constantly.

DEF The **reference angle** $\hat{\theta}$ is the acute angle between the arm OP and the x-axis. It is positive, it is less than $\frac{\pi}{2}$, and it ignores signs entirely: you get it from the size of k alone, as $\arcsin|k|$, $\arccos|k|$ or $\arctan|k|$. It is defined whenever the arm lies off the axes.
The **principal value** is the single angle your calculator returns. It is the one solution lying inside the range of the matching inverse function:

$$\arcsin k \in \left[-\tfrac{\pi}{2},\, \tfrac{\pi}{2}\right], \qquad \arccos k \in [0,\, \pi], \qquad \arctan k \in \left(-\tfrac{\pi}{2},\, \tfrac{\pi}{2}\right).$$

The two are easy to confuse. A principal value can be negative, and for a cosine it can be obtuse; a reference angle is never either. Take $\cos x = -\frac{1}{2}$. The calculator returns the principal value $\frac{2\pi}{3}$, while the reference angle is $\frac{\pi}{3}$, the acute angle the arm makes with the negative x-axis. Work from the reference angle and the calculator's choice of quadrant stops mattering.

The method. Every solution is built from the reference angle and the quadrant it sits in. The sign of k tells you which two quadrants to use, by the rule of §8.1.2, and each quadrant has one construction.

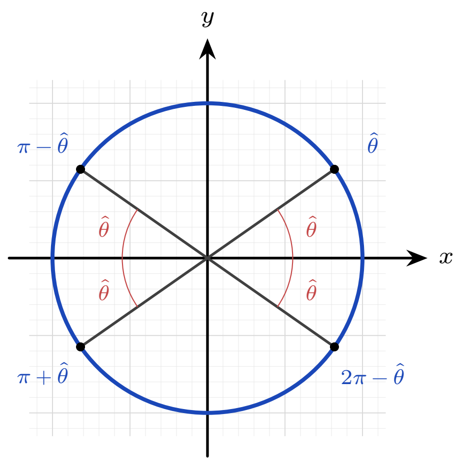

Figure 8.9 One reference angle $\hat{\theta}$, measured from the x-axis, gives an angle in each quadrant. The scarlet arcs are all the same size; only the direction they are measured from changes.

KEY **Solving a trigonometric equation.**

1. Rearrange until a single trigonometric function is alone on one side.
2. If the argument is not plain x, substitute u for it and **transform the interval as well**.
3. Find the reference angle from the *size* of k: $\hat{\theta} = \arcsin|k|$, $\arccos|k|$ or $\arctan|k|$.
4. Use the sign of k to choose the two quadrants, then build the angle in each: $\hat{\theta}$, $\pi - \hat{\theta}$, $\pi + \hat{\theta}$ or $2\pi - \hat{\theta}$.
5. Add or subtract whole turns of 2π until you have every solution the interval asks for.
6. If you substituted, convert back to x at the very end.

There is a shorter route once you already have one solution, which you do whenever a solution exists (for sine and cosine, when $|k| \le 1$): the calculator has just given you one. Call it θ. Its partner in the same turn is fixed by the function alone, whatever quadrant θ landed in.

KEY **The partner solution.** If θ is one solution, then the other in the same turn is

$$\sin x = k: \quad \pi - \theta, \qquad \cos x = k: \quad 2\pi - \theta, \qquad \tan x = k: \quad \pi + \theta.$$

Then add or subtract whole turns of 2π (for the tangent, half turns of π) to reach every solution the interval asks for.

The interval decides how the partner is written, not what it is. On $-\pi \le x \le \pi$ the cosine pair is θ and $-\theta$, because $2\pi - \theta$ is one whole turn away from $-\theta$. That is why a cosine equation on a symmetric interval always answers $\pm$ the same number, and it is worth recognising on sight: $\cos x = \frac{1}{2}$ on $-\pi \le x \le \pi$ gives $x = \pm\frac{\pi}{3}$ and nothing else.

Both routes give the same angles. The reference angle is the one to fall back on when the signs get confusing, or when the equation has been rearranged so far that you no longer trust which quadrant you are in.

Two checks belong in every solution. Your two angles should lie in the quadrants where that function has the sign of k; if only one does, you have built the wrong pair. And when $k = \pm 1$ the two constructions land on the same point of the circle, so report that angle once in each turn. Keep both ends when the interval includes them: $\cos x = 1$ on $0 \le x \le 2\pi$ gives $x = 0$ and $x = 2\pi$.

Take care at steps 2 and 5: a solution missed there does not reappear later.

Worked through. To solve $\cos x = -\frac{1}{2}$ for $0 \le x \le 2\pi$, take the reference angle from the size of k alone:

$$\hat{\theta} = \arccos\left|-\tfrac{1}{2}\right| = \arccos\tfrac{1}{2} = \frac{\pi}{3}.$$

Cosine is negative in the second and third quadrants, so build one angle in each:

$$\pi - \frac{\pi}{3} = \frac{2\pi}{3}, \qquad \pi + \frac{\pi}{3} = \frac{4\pi}{3}.$$

Both lie in the interval, so $x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$. Notice that the calculator alone would have given only $\frac{2\pi}{3}$. The reference angle gives both, and the sign of k says where they live. The same steps work when k is not an exact value: only the reference angle now comes from the calculator.

TIP On a calculator paper, always check your solution count graphically: plot both sides and count the intersections in the interval. If your algebra found three solutions and the screen shows four, look for the one you missed.

When the argument is not plain x but $2x$ or $3(x-4)$, substitute for the whole argument and, crucially, transform the interval along with it. A doubled argument sweeps the circle twice as fast, so it collects twice as many solutions.

EXAMPLE 8.4

Solve $\cos 2x = \dfrac{\sqrt{3}}{2}$ for $0 \le x \le 2\pi$.

Let $u = 2x$. As x runs over $[0, 2\pi]$, u runs over $[0, 4\pi]$: two full turns.

The reference angle is $\hat{\theta} = \arccos\frac{\sqrt{3}}{2} = \frac{\pi}{6}$. Cosine is positive, so the first turn gives

$$u = \frac{\pi}{6} \quad \text{and} \quad u = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}.$$

The interval covers a second turn, so add 2π to each:

$$\frac{\pi}{6} + 2\pi = \frac{13\pi}{6}, \qquad \frac{11\pi}{6} + 2\pi = \frac{23\pi}{6}.$$

A third turn would pass 4π, so the list is complete:

$$u = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6} \implies x = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}.$$

Four solutions, not two. Note where the doubling acts: the step in u is still 2π, because $\cos u$ has period 2π whatever u came from. What doubling changes is the *interval*, and that is why two extra solutions appear. In x the period has become π.

8.6.1 When the interval is not 0 to 2π

The interval is part of the question, and it is not always $0 \le x \le 2\pi$. Another interval you will meet is $-\pi \le x \le \pi$. That is still one whole turn, but measured differently: half a turn clockwise from the positive x-axis, and half a turn anticlockwise.

The method does not change. Build the two angles from the reference angle exactly as before, then shift either one by a whole turn if it falls outside the interval: subtract 2π from an angle above π, add 2π to an angle below $-\pi$. A whole turn returns the point to exactly where it was, so the value of the function is untouched.

EXAMPLE 8.5

Solve $\cos x = -\dfrac{1}{2}$ for $-\pi \le x \le \pi$.

The reference angle is $\hat{\theta} = \arccos \frac{1}{2} = \frac{\pi}{3}$. Cosine is negative, so the two quadrants are the second and the third:

$$\pi - \frac{\pi}{3} = \frac{2\pi}{3}, \qquad \pi + \frac{\pi}{3} = \frac{4\pi}{3}.$$

The first already lies in $[-\pi,\ \pi]$. The second does not, because $\frac{4\pi}{3} > \pi$, so subtract one turn:

$$\frac{4\pi}{3} - 2\pi = -\frac{2\pi}{3}.$$

So $x = \frac{2\pi}{3}$ or $x = -\frac{2\pi}{3}$.

The two answers form a $\pm$ pair, and that is not a coincidence. Cosine is even, so on any interval symmetric about 0 its answers are $\pm\alpha$ for a single angle α. Sine is odd, and its pair on the same interval is

$$\hat{\theta} \text{ and } \pi - \hat{\theta} \quad \text{when } k > 0, \qquad -\hat{\theta} \text{ and } -\pi + \hat{\theta} \quad \text{when } k < 0.$$

8.6.2 Using identities

An equation mixing $\sin 2x$ with $\sin x$, or $\cos 2x$ with $\cos x$, cannot be solved as it stands. An identity rewrites everything in terms of a single function, turning the equation into an algebraic one, usually a quadratic.

Worked through. To solve $\cos 2x + 3\cos x + 2 = 0$ for $0 \le x \le 2\pi$, choose the form of $\cos 2x$ that gives only cosines, $\cos 2x = 2\cos^2 x - 1$:

$$2\cos^2 x - 1 + 3\cos x + 2 = 0 \implies 2\cos^2 x + 3\cos x + 1 = 0.$$

This factors as $(2\cos x + 1)(\cos x + 1) = 0$, so $\cos x = -\frac{1}{2}$ or $\cos x = -1$. From the circle,

$$x = \frac{2\pi}{3}, \frac{4\pi}{3} \quad \text{or} \quad x = \pi.$$

Picking the form of the double-angle identity that matches the other terms is the step that turns these problems into quadratics.

YOUR TURN 8.6 Solving Trigonometric Equations

Exact solutions

36. Solve for $0 \le x \le 2\pi$.

(a) $\sin x = \dfrac{1}{2}$

(b) $\cos x = 0$

(c) $\tan x = 1$

(d) $\cos x = 1$

37. Solve for $0 \le x \le 2\pi$.

(a) $\sin x = -\dfrac{1}{2}$

(b) $2\cos x = \sqrt{3}$

(c) $\sqrt{2}\sin x = -1$

(d) $\tan x = -\sqrt{3}$

Other intervals

38. Solve for $-\pi \le x \le \pi$. Give every answer inside that interval.

(a) $\cos x = \dfrac{\sqrt{3}}{2}$

(b) $\sin x = -\dfrac{1}{2}$

(c) $\tan x = -1$

(d) $2\sin x = \sqrt{2}$

Answers that are not exact

39. GDC Solve for $0 \le x \le 2\pi$, giving your answers to three significant figures. If there is no solution, say why.

(a) $\cos x = 0.3$

(b) $3\sin x = -2$

(c) $4\cos x + 5 = 0$

40. GDC Solve $\sin x = -0.8$ for $0 \le x \le 2\pi$, giving your answers to three significant figures.

Multiple angles

41. Solve for $0 \le x \le 2\pi$. Transform the interval before solving.

(a) $\sin 2x = 0$

(b) $\cos 2x = \dfrac{1}{2}$

(c) $\tan 2x = 1$

Shifted arguments

42. Solve for $0 \le x \le 2\pi$. Transform the interval before solving.

(a) $\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$

(b) $\cos\left(x - \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

(c) $2\sin\left(2x - \frac{\pi}{3}\right) = \sqrt{3}$

Factorising

43. Factorise, then solve for $0 \le x \le 2\pi$.

(a) $2\sin^2 x - \sin x = 0$

(b) $2\cos^2 x + \cos x - 1 = 0$

(c) $2\sin^2 x + 3\sin x + 1 = 0$

(d) $\tan^2 x - \tan x = 0$

Using an identity first

44. Use an identity to rewrite each equation, then solve for $0 \le x \le 2\pi$.

(a) $\cos 2x + \sin x = 0$

(b) $\sin 2x = \cos x$

(c) Explain why dividing both sides of the equation in (b) by $\cos x$ would lose solutions.

(d) $2\sin^2 x = 3\cos x$

Concept check

45. To solve $\sin 2x = \frac{1}{2}$ for $0 \le x \le 2\pi$, a student writes: "$2x = \frac{\pi}{6}$ *or* $\frac{5\pi}{6}$, *so* $x = \frac{\pi}{12}$ *or* $\frac{5\pi}{12}$." Identify the error and give the full solution set.

Chapter 8 · Trigonometry

Reflections

TOK A sine wave repeats forever, identical in every cycle, and nothing in the physical world does that exactly. Yet sums of sine waves describe voices, tides and heartbeats closely enough to compress, predict and diagnose them. Is the sinusoid a human tool, or an ‘alphabet’ of nature, a basic set of shapes from which the repeating patterns of the physical world are built?

IM Why is a full turn $360°$? The number comes from Babylonian astronomy, which counted in base 60, and 360 divides exactly by 2, 3, 4, 5, 6, 8, 9, 10 and 12. During the French Revolution, the reformers who introduced the metre also proposed a decimal angle, the *grade*, with 400 in a full turn and 100 in a right angle. Most of the world kept degrees, but the grade, now called the *gon*, is still used by surveyors in parts of continental Europe, and many calculators still offer a GRAD mode. Which unit is right depends on who is measuring and why, and calculus, as Chapter 13 shows, uses neither: it uses radians.

RW Because a sine wave is what steady rotation looks like from the side, it models many oscillations in science and engineering. Alternating current rises and falls as a sine while the generator spins. Sound is pressure oscillating in the same way, and its pitch is the frequency of the wave. Light and radio are sinusoids too, and the signals your phone sends and receives are built from them. Fourier analysis, decomposing a signal into its sine components, is used to compress music and images and to read earthquake and heart signals.

EXT In radians, $\sin x$ and x almost agree for small x: $\sin 0.1 = 0.09983\ldots$ On the unit circle x is the arc and $\sin x$ is the height, and over a tiny arc there is almost no difference between them. Physicists use this as the **small-angle approximation**: for $|x|$ small and in radians,

$$\sin x \approx x, \qquad \tan x \approx x, \qquad \cos x \approx 1 - \tfrac{1}{2}x^2.$$

At $x = 0.1$ these give 0.1, 0.1 and 0.995, against true values 0.09983, 0.10033 and 0.99500. The pendulum equation can be solved in elementary functions only when $\sin\theta$ is replaced by θ, which is why the simple period formula holds for small swings and not for large ones. All three fail in degrees, and that is one reason calculus uses radians. Chapter 13 turns the first into $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, the limit that gives $\dfrac{d}{dx}\sin x = \cos x$.

End-of-Chapter Problems
Chapter 8: Trigonometry

1. Use the unit circle to write down the exact value of each.

(a) $\sin\dfrac{2\pi}{3}$ [1] (b) $\cos\dfrac{2\pi}{3}$ [1]

(c) $\tan\dfrac{2\pi}{3}$ [1] (d) $\cos\dfrac{5\pi}{4}$ [1]

(e) $\sin\dfrac{11\pi}{6}$ [1] (f) $\tan\dfrac{3\pi}{4}$ [1]

2. The point P lies on the unit circle at angle θ.

(a) State the coordinates of P when $\theta = \dfrac{5\pi}{6}$. [2]

(b) P has coordinates $\left(-\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$. Find θ for $0 \le \theta \le 2\pi$. [2]

(c) P lies in the third quadrant. State the sign of $\sin\theta$, of $\cos\theta$ and of $\tan\theta$. [2]

(d) Given $\sin\theta = \dfrac{\sqrt{3}}{2}$, write down the two possible values of θ in $0 \le \theta \le 2\pi$, and explain why there are exactly two. [3]

3. Use the symmetry of the unit circle to answer each part.

(a) Given $\sin 0.6 = 0.565$, write down $\sin(\pi - 0.6)$. [1]

(b) Given $\cos 1.2 = 0.362$, write down $\cos(-1.2)$ and $\cos(2\pi - 1.2)$. [2]

(c) Explain why $\sin(\theta + 2\pi) = \sin\theta$ for every θ. [2]

(d) Explain why $\tan(\theta + \pi) = \tan\theta$, but $\sin(\theta + \pi) \ne \sin\theta$. [3]

4. Consider $f(x) = 3\sin(2x) + 1$.

(a) State the amplitude and the period. [2]

(b) State the maximum and minimum values. [2]

5. The graph of $y = a\sin(bx) + d$ has maximum 7, minimum 1, and period π. Find a, b and d. [4]

6. Sketch $y = \cos x - 1$ for $0 \le x \le 2\pi$, stating the maximum, the minimum, and the intercepts. [4]

7. The depth of water in a harbour is $d(t) = 5 + 3\sin\left(\dfrac{\pi t}{6}\right)$ metres, t hours after midnight.

(a) State the maximum and minimum depths. [2]

(b) Find the period, and say what it represents. [2]

8. A seat on a Ferris wheel has height $h(t) = 10 - 8\cos\left(\frac{\pi t}{15}\right)$ metres, t seconds after the start.

(a) State the minimum and maximum heights. [2]
(b) Find the time for one full revolution. [2]

9. Evaluate, giving exact values.

(a) $\arctan 1$ [1]
(b) $\arcsin\left(-\frac{1}{2}\right)$ [1]
(c) $\arccos(-1)$ [1]
(d) $\arcsin\left(\sin\frac{4\pi}{3}\right)$ [2]

10. Find the exact value of each.

(a) $\arccos\left(\cos\frac{7\pi}{4}\right)$ [2]
(b) $\tan\left(\arcsin\frac{8}{17}\right)$ [2]
(c) $\cos\left(\arcsin\frac{3}{5} + \arctan\frac{4}{3}\right)$, and hence $\arcsin\frac{3}{5} + \arctan\frac{4}{3}$ [4]
(d) $\tan\left(2\arctan\frac{1}{3}\right)$ [3]

11. Given $\sin\theta = \frac{5}{13}$ with θ in the second quadrant, find the exact value of each.

(a) $\cos\theta$ [2]
(b) $\tan\theta$ [1]
(c) $\sin 2\theta$ [2]
(d) $\cos 2\theta$ [2]

12. Given $\tan\theta = 2$ with θ acute, find the exact value of each.

(a) $\sin\theta$ and $\cos\theta$ [2]
(b) $\sin 2\theta$ [2]
(c) $\cos 2\theta$ [2]

13. Use a compound-angle formula to find each exact value.

(a) $\tan 15^\circ$ [3]
(b) $\sin 15^\circ$ [2]
(c) $\sin 105^\circ$ [2]
(d) $\tan 105^\circ$ [3]

14. Solve $\sec^2 x = 4$ for $0 \le x < \frac{\pi}{2}$. [2]

15. Solve for $0 \le x \le 2\pi$.

(a) $2\cos x + 1 = 0$ [2]
(b) $\sqrt{3}\tan x = 1$ [2]

(c) $2\cos^2 x = 1$ [3] (d) $\tan^2 x = 3$ [3]

16. Solve each equation for $-\pi \leq x \leq \pi$. Build every answer from the reference angle, then shift by a whole turn where the interval requires it.

(a) $\sqrt{3}\tan x = -1$ [3] (b) $\sin 2x = \dfrac{\sqrt{3}}{2}$ [4]

(c) $\tan\left(x + \dfrac{\pi}{4}\right) = \sqrt{3}$ [4]

17. Solve each equation on the interval given. Take care: the same equation has different answers on different intervals.

(a) $\cos x = -\dfrac{\sqrt{2}}{2}$ for $0 \leq x \leq 2\pi$, and then for $-\pi \leq x \leq \pi$. [4]

(b) $\sin x = -\dfrac{\sqrt{3}}{2}$ for $-\pi \leq x \leq \pi$. [3]

(c) Explain why an equation of the form $\cos x = k$, with $|k| < 1$, always has two solutions on $-\pi \leq x \leq \pi$, and why they are negatives of each other. [3]

18. Solve for $0 \leq x \leq 2\pi$. Each becomes a quadratic.

(a) $2\sin^2 x + \sin x - 1 = 0$ [4] (b) $4\sin^2 x - 3 = 0$ [3]

(c) $\cos 2x = \cos x$ [4] (d) $\sin 2x = \sin x$ [4]

19. Solve for $0 \leq x \leq 2\pi$.

(a) $2\sin x\cos x = \dfrac{1}{2}$ [3]

(b) $\cos 2x + 3\sin x = 2$ [5]

(c) $\sin x + \cos x = 1$ [5]

20.

(a) Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [5]

(b) Hence solve $8\cos^3\theta - 6\cos\theta = 1$ for $0 \leq \theta \leq \pi$. [4]

21. In this question, θ takes only values for which every expression is defined.

(a) Prove that $\dfrac{1+\sin 2\theta}{\cos 2\theta} = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$. [4]

(b) Show that $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \tan\left(\theta + \dfrac{\pi}{4}\right)$. [3]

(c) Hence solve $\dfrac{1+\sin 2\theta}{\cos 2\theta} = \sqrt{3}$ for $0 \leq \theta \leq 2\pi$. [4]

22. In this question, θ takes only values for which every expression is defined.

(a) Prove that $\cot\theta - \tan\theta = 2\cot 2\theta$. [4]

(b) Prove that $\operatorname{cosec}\theta - \cot\theta = \tan\frac{\theta}{2}$. [5]

(c) Use part (a) with $\theta = \frac{\pi}{8}$ to show that $\tan\frac{\pi}{8} = \sqrt{2} - 1$, and check the result with part (b). [4]

23. GDC

(a) Show that $3\sin x + 4\cos x$ can be written as $5\sin(x+\alpha)$, where $\tan\alpha = \frac{4}{3}$ and $0 < \alpha < \frac{\pi}{2}$. Find α. [3]

(b) Hence write down the maximum value of $3\sin x + 4\cos x$, and find the smallest positive value of x at which it occurs. [2]

24. GDC Let $g(x) = 5\sin x + 12\cos x$ for $0 \le x \le 2\pi$.

(a) Write $g(x)$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. [3]

(b) Hence find the maximum value of g, and the value of x at which it occurs. [2]

(c) Solve $g(x) = 6.5$. [4]

25. GDC Let $f(x) = 2\sin(2x) + 1$ and $g(x) = \frac{x}{2}$, for $0 \le x \le 2\pi$.

(a) Write down the period and the range of f. [2]

(b) Find the x-coordinates of the points where the graphs of f and g meet. [4]

(c) Hence solve $f(x) > g(x)$. [3]

(d) Find the greatest value of $f(x) - g(x)$, and the value of x at which it occurs. [2]

26. GDC The mean monthly temperature in a coastal city is modelled by

$$T(t) = a\cos\left(b(t-8)\right) + d,$$

where T is in °C, t is the time in months from the start of the year, and $a, b > 0$. The model has a maximum of 27 °C at $t = 8$, and its next minimum is 5 °C at $t = 14$.

(a) Find a and d. [2]

(b) Find b. [2]

(c) Find $T(4)$. [2]

(d) Solve $T(t) > 20$ for $0 \le t \le 12$, and hence find the length of time in each year for which the model gives a mean temperature above 20 °C. [5]

27. GDC **Daylight in Sapporo.** The number of hours of daylight is modelled by

$$L(t) = 12.2 + 3.4\sin\left(\frac{2\pi(t-81)}{365}\right),$$

where t is the day of the year, with $t = 1$ on 1 January.

(a) State the maximum and minimum daylight, and find the value of t at which each occurs. [4]

(b) State the period of L, and say what it represents. [2]

(c) Find the daylight on 1 January, to three significant figures. [2]

(d) Find the two days of the year on which the daylight is exactly 12.2 hours, and explain what is special about them. [4]

(e) Find the values of t for which $L(t) \geq 15$, and state how many days of the year have at least 15 hours of daylight. [6]

28. **One function, two forms.** Let $f(x) = 2\sin x + \cos 2x$ for $0 \leq x \leq 2\pi$.

(a) Show that $f(x) = -2\sin^2 x + 2\sin x + 1$. [3]

(b) Let $u = \sin x$. Write f as a quadratic in u, and state the interval of values u can take. [3]

(c) Hence show that the maximum value of f is $\dfrac{3}{2}$. [4]

(d) Find the values of x in $[0, 2\pi]$ at which that maximum occurs. [3]

(e) Solve $f(x) = 1$ for $0 \leq x \leq 2\pi$. [5]

(f) State the range of f. [3]

Challenge Questions

29. GDC **When is a sum of waves periodic?** A function f is *periodic* with period $T > 0$ if $f(x+T) = f(x)$ for every x. Every sine wave is periodic, but a sum of two sine waves need not be. This investigation finds when the sum repeats, what it looks like when the two periods are close, and what happens when it never repeats.

(a) State the periods of $\sin 2x$ and $\sin 3x$. Show that $f(x) = \sin 2x + \sin 3x$ satisfies $f(x + 2\pi) = f(x)$, and by evaluating $f(\frac{\pi}{2})$ and $f(\frac{3\pi}{2})$ show that π is not a period of f. [3]

(b) Let p and q be positive numbers with $\dfrac{p}{q} = \dfrac{m}{n}$, where m and n are positive integers. Show that $\sin px + \sin qx$ is periodic, with period $\dfrac{2\pi m}{p}$, and check this against part (a). [3]

(c) By writing $A = u + v$ and $B = u - v$ and expanding $\cos(u+v) + \cos(u-v)$, show that

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).$$

[3]

(d) Two tuning forks sound at 440 Hz and 442 Hz together. The combined air pressure is modelled by $y = \cos(880\pi t) + \cos(884\pi t)$, with t in seconds. Write y as a product, and use it to describe the graph of y. Hence find how many times each second the sound is loudest, and state the general rule for forks of frequencies f_1 and f_2 Hz that are close together. [4]

(e) Let $g(x) = \cos x + \cos(\sqrt{2}\,x)$. Show that $g(x) = 2$ only when $x = 0$, and deduce that g is not periodic. You may assume that $\sqrt{2}$ is irrational. [5]

(f) Use technology to find the greatest value of $g(x)$ for $5 \le x \le 100$, and the value of x at which it occurs. Explain why this x is close to a multiple of 2π, by relating it to a fraction close to $\sqrt{2}$. Suggest how this could be investigated further. [3]

30. $\cos n\theta$ **as a polynomial in** $\cos\theta$**.** You have already met $\cos 2\theta = 2\cos^2\theta - 1$ and, in the end-of-chapter problems, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. Each is a polynomial in $\cos\theta$ alone. This investigation shows that this always happens, and finds the pattern.

Write $c = \cos\theta$ throughout.

(a) Use the compound-angle formulas to show that

$$\cos((n+1)\theta) = 2\cos\theta\,\cos(n\theta) - \cos((n-1)\theta).$$

(Hint: expand $\cos(n\theta + \theta)$ and $\cos(n\theta - \theta)$, then add) [5]

(b) Starting from $\cos 0 = 1$ and $\cos\theta = c$, use the recurrence to obtain $\cos 2\theta$ and $\cos 3\theta$ in terms of c. [3]

(c) Use it again to find $\cos 4\theta$ and $\cos 5\theta$ in terms of c. [4]

(d) For $n \ge 1$, state the degree of the polynomial for $\cos n\theta$, and the coefficient of its highest power. Justify both from the recurrence. [4]

(e) Explain why no such polynomial in $\cos\theta$ alone exists for $\sin n\theta$, for any $n \ge 1$. *(Hint: compare $\sin(-\theta)$ with $\sin\theta$)* [3]

(f) Find the three roots of $4c^3 - 3c = 0$ by solving $\cos 3\theta = 0$ for $0 \le \theta \le \pi$. Generalise: state the n roots of the polynomial for $\cos n\theta$, explain why there are no others, and suggest how this could be investigated further. [3]

31. GDC **The window of safe passage.** At a harbour mouth the depth of water is modelled by

$$d(t) = 6 + 2\sin\left(\frac{\pi t}{6}\right) \text{ metres},$$

where t is the number of hours after midnight. A ferry can enter only while the depth is at least the figure its hull requires. This investigation finds the window of time available, and then finds it in general.

(a) State the greatest and least depths, the period of the tide, and the time of the first high tide. [4]

(b) The ferry needs at least 7 m of water. Solve $d(t) \ge 7$ on $0 \le t \le 12$, and state the window as a pair of clock times. [5]

(c) Show that the window is exactly 4 hours long, and explain why the same window opens once in every tidal cycle. [3]

(d) Now take a general tide $d(t) = m + a\sin\left(\frac{2\pi t}{T}\right)$ and a required depth k with $m < k < m + a$. Show that the time available in each cycle is

$$\frac{T}{\pi}\arccos\left(\frac{k-m}{a}\right).$$

(Hint: solve for the two ends in terms of $\arcsin$*, subtract, then use* $\frac{\pi}{2} - \arcsin c = \arccos c$*)* [6]

(e) A larger ferry needs 7.5 m. Use the formula to find its window, to the nearest minute. Then explain, from the shape of the arccosine graph, why the last half-metre of clearance costs far more time than the first. [4]

(f) Real tides are closer to a sum of waves. The two largest have periods of about 12.42 hours (from the Moon) and 12.00 hours (from the Sun), so a better model is

$$d(t) = 6 + 1.5\sin\left(\frac{2\pi t}{12.42}\right) + 0.5\sin\left(\frac{2\pi t}{12}\right).$$

Explain why the range between high and low water grows and shrinks, state the largest and smallest ranges, and find, in days, how long the range takes to return to its largest value. Suggest how the window of part (d) could be studied for this model. [3]

32. GDC **Machin's formula for π.** In 1706 John Machin used the compound-angle formula for the tangent to write π in terms of two small arctangents, and so found 100 decimal places of π by hand. This investigation proves his formula and shows why small arguments help.

(a) Use the formula for $\tan(A + B)$ to show that $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$. Explain why the sum on the left must lie between 0 and $\frac{\pi}{2}$. [3]

(b) Let $\alpha = \arctan\frac{1}{5}$. Show that $\tan 2\alpha = \frac{5}{12}$ and $\tan 4\alpha = \frac{120}{119}$. [3]

(c) Show that $\tan(4\alpha - \frac{\pi}{4}) = \frac{1}{239}$. [2]

(d) Show that $0 < 4\alpha - \frac{\pi}{4} < \frac{\pi}{2}$, and hence prove Machin's formula

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

(Hint: use part (b) to show first that $2\alpha < \frac{\pi}{4}$*)* [4]

(e) For small x, $\arctan x \approx x - \dfrac{x^3}{3} + \dfrac{x^5}{5}$ (you will meet this series in Ch. 16). Use these three terms for $\arctan\frac{1}{5}$, and only the first term for $\arctan\frac{1}{239}$, to estimate π, and compare your estimate with the value of π. Explain why Machin's formula gives π far faster than the formula of part (a) would. [4]

(f) Show that for every positive integer n,

$$\arctan\frac{1}{n} - \arctan\frac{1}{n+1} = \arctan\frac{1}{n^2+n+1},$$

and check that $n = 1$ gives part (a). Suggest how this could be used to investigate other formulas of Machin's kind. [3]