CHAPTER 7

Shapes & Measures

7

SYLLABUS 3.1 3.2 3.3 3.4

You cannot pace out the whole Earth. So how did anyone find its size? The answer came from something much smaller: an angle.

Check the fact yourself. On flat ground the horizon is level with your eye. Climb a hill and it drops slightly below your eye line, and the higher you climb, the further it drops. That drop is the **dip**, and it is small: about half a degree from a hill a few hundred metres high, the width of the Moon in the sky.

Your line of sight to the horizon is a tangent to the Earth: it touches the surface at one point, and there it meets the radius at a right angle. So the hilltop, the horizon and the centre of the Earth form one right-angled triangle, with hill height h and dip θ:

$$\cos\theta = \frac{R}{R+h}, \qquad \text{so} \qquad R = \frac{h\cos\theta}{1-\cos\theta}.$$

Measure the hill, measure the dip, and the radius comes out close to the modern 6371 km, from one hill, in one afternoon. Al-Bīrūnī did exactly this a thousand years ago, from the fort of Nandana in what is now Pakistan.

Size matters as well as shape. Double every length of an object and its surface grows fourfold while its volume grows eightfold, so bigger things have less surface for what they hold. That is why crushed ice melts faster than a block, and it is one reason insects do not grow to the size of a horse.

By the end of this chapter you will find the angle between a line and a plane in a solid, work out the volume and surface area of standard solids, solve any triangle with the sine and cosine rules, ambiguous case included, handle elevation, depression and bearings, and measure in **radians**, which turn arc length and sector area into one multiplication each.

> A calculator is assumed throughout this chapter. The EXACT badge marks the questions whose answer must be left exact.

7.1 Three-Dimensional Geometry

The measurements you need from a solid are often not the ones marked on its faces. The diagonal of a room, or the angle a roof strut makes with the floor: neither of these lies along a face. Both run through the inside of the object. Right-angled triangles drawn inside the solid, and sometimes coordinates, are how you reach them.

Before any measuring, a word on names. Examination papers use a fixed notation for points, segments, lines and angles, and they use it without explanation.

Notation	Meaning
$A(x, y)$	the point A with coordinates x and y
$[AB]$	the line segment with endpoints A and B
AB	the length of $[AB]$
(AB)	the line through A and B, continuing both ways
$A\hat{B}C$	the angle at B, between $[BA]$ and $[BC]$
$\triangle ABC$	the triangle with vertices A, B and C

The middle letter names the vertex. So $A\hat{B}C$ and $C\hat{B}A$ are the same angle, while $B\hat{A}C$ is a different one. Note too that the length of a segment carries no bars. Bars are reserved for the magnitude of a vector, which Chapter 9 introduces.

Coordinates in the plane extend to space by adding a third axis, z. The distance formula extends with them. In two dimensions it is Pythagoras' theorem. In three dimensions it is Pythagoras used *twice*, as shown below the formula.

> KEY · IB The distance between (x_1, y_1, z_1) and (x_2, y_2, z_2) is
>
> $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$
>
> The midpoint is the average of the coordinates, $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$.

Across the base, Pythagoras gives

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Treat that as one side of a new right triangle. Its other side is the vertical rise, $(z_2 - z_1)$, and Pythagoras on this second triangle gives the formula.

The equation of a sphere comes from the same formula. A sphere is the set of points at a fixed distance r from a centre (a, b, c), so its equation is $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$. The sphere equation is beyond the syllabus, but it follows from the distance formula in one line.

7.1.1 The angle between a line and a plane

A line that is not parallel to a plane meets it at an angle. Many angles can be measured at that meeting point, so one of them has to be chosen, and mathematics chooses the smallest. Drop a perpendicular from each point of the line to the plane. The feet of those perpendiculars form a line in the plane, called the **projection** of the line onto the plane (its 'shadow': what a light directly overhead would cast). The angle between the line and its projection is the smallest angle available.

> DEF The **angle between a line and a plane** is the angle between the line and its projection on the plane. To find it, drop a perpendicular from a point on the line down to the plane. The line, its projection and that perpendicular form a right triangle, and right-angled trigonometry solves it.

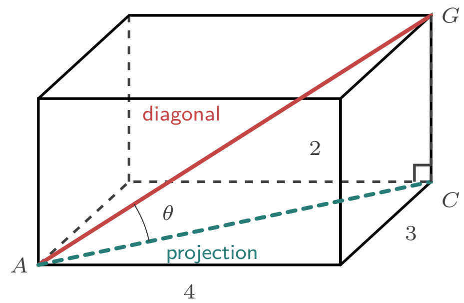

Figure 7.1 The space diagonal AG and its projection AC on the base. The vertical edge CG meets the base at a right angle, so ACG is a right triangle and θ is the angle between the diagonal and the base.

Worked through. To find the angle between the space diagonal AG and the base of the cuboid in the figure, with its 4×3 base and height 2, start with the diagonal of the base: $AC = \sqrt{4^2 + 3^2} = 5$. This is the projection of AG onto the base, and $CG = 2$ is vertical, meeting the base at a right angle. In the right triangle ACG,

$$\tan\theta = \frac{CG}{AC} = \frac{2}{5} \implies \theta = \arctan(0.4) \approx 21.8^\circ.$$

The order is the same each time. Find the projection on the plane, then the perpendicular height, then use $\tan$.

7.1.2 The angle between two intersecting lines

Two lines that meet also make an angle, and inside a solid those two lines are often edges from the same corner. Take the two edges, then join their far ends. The three segments form a triangle, and the angle you want sits at the corner. Measure the three sides with the distance formula, then find the angle with the cosine rule of §7.3. When the triangle is isosceles, as in the case worked below, halving it gives a right triangle instead.

Worked through. A square-based pyramid has base corners $A(0,0,0)$, $B(6,0,0)$, $C(6,6,0)$, $D(0,6,0)$ and apex $V(3,3,4)$. To find the angle between the slant edges VA and VB, note that the two edges meet at V and span the triangle VAB, so the angle we want is $A\hat{V}B$. First measure the triangle. By the distance formula,

$$VA = \sqrt{(3-0)^2 + (3-0)^2 + (4-0)^2} = \sqrt{34},$$

and $VB = \sqrt{34}$ by the same calculation, while $AB = 6$ along the base.

The triangle is isosceles, so cut it in half. The midpoint of AB is $M(3,0,0)$, and VM meets AB at a right angle. In the right triangle VMA, the side $AM = 3$ is opposite the half-angle at V, and the hypotenuse is $VA = \sqrt{34}$:

$$\sin\frac{\theta}{2} = \frac{3}{\sqrt{34}} \implies \frac{\theta}{2} \approx 31.0^\circ \implies \theta \approx 61.9^\circ.$$

This uses the two formulas of this section: the distance formula for the three sides, the midpoint to cut the triangle in half, then right-angled trigonometry for the angle.

YOUR TURN 7.1 Three-Dimensional Geometry

Distance and midpoint

1. Find the distance between each pair of points.

(a) $(0,0,0)$ and $(2,3,6)$

(b) $(2,-1,0)$ and $(-2,3,7)$

(c) $(-3,4,1)$ and $(3,1,-1)$

(d) $(1.5,0,-2)$ and $(-0.5,4,2)$

2. Answer each part.

(a) Find the midpoint of $(1,2,3)$ and $(5,4,1)$.

(b) Find the midpoint of $(2,-1,4)$ and $(6,3,0)$.

(c) $M(2,-1,5)$ is the midpoint of $[AB]$, where A is $(-1,3,4)$. Find the coordinates of B.

Angle between a line and a plane

3. A cuboid has base 3×4 and height 12.

(a) Find the length of the diagonal of the base.
(b) Find the length of the space diagonal.
(c) Find the angle between the space diagonal and the base.

4. A square-based pyramid has base side 6, and its apex is 5 vertically above the centre of the base.

(a) Find the distance from a corner of the base to the centre of the base.
(b) Find the angle between a slant edge and the base.

Angle between two lines

5. The points $O(0,0,0)$, $P(4,3,0)$ and $Q(0,3,2)$ are three corners of a cuboid.

(a) Find the lengths OP, OQ and PQ.
(b) Use the cosine rule to find the angle $P\hat{O}Q$ between the lines (OP) and (OQ).

6. A pyramid has square base $A(0,0,0)$, $B(4,0,0)$, $C(4,4,0)$, $D(0,4,0)$ and apex $V(2,2,6)$.

(a) Show that $VA = VB = 2\sqrt{11}$.
(b) Find the angle $A\hat{V}B$ between the slant edges VA and VB. *(Hint: the midpoint of $[AB]$ cuts triangle VAB into two right triangles)*

Concept check

7. A cuboid has a base 6 by 8 and height 5. To find the angle between a space diagonal and the base, a student writes $\tan\theta = \dfrac{5}{8}$, so $\theta \approx 32.0^\circ$.

(a) Identify the error.
(b) Find the correct angle.

7.2 Volumes and Surface Areas of Solids

You already know the cuboid, the prism and the cylinder. The formula booklet adds the sphere, the cone and the pyramid. Knowing where each formula comes from makes it harder to misapply.

KEY · IB

Sphere, radius r	$V = \frac{4}{3}\pi r^3$, $\quad A = 4\pi r^2$
Cone, radius r, height h	$V = \frac{1}{3}\pi r^2 h$, $\quad$ curved $A = \pi r l$
Pyramid, base area B, height h	$V = \frac{1}{3}Bh$
Cylinder, radius r, height h	$V = \pi r^2 h$, $\quad$ curved $A = 2\pi r h$

The cylinder is listed in the *Prior learning* table at the front of the booklet rather than beside the other three.

Each formula has a short reason behind it.

- **Cylinder.** Cut the curved side along one vertical line and unroll it into a rectangle of height h and width $2\pi r$, the base circumference. Its area is $2\pi r h$.
- **Cone.** The same cut unrolls the curved side into a sector. Its radius is the **slant height** l, measured from the rim to the tip, and its arc is the base circumference $2\pi r$. A sector's area is half its arc times its radius, so the curved area is $\frac{1}{2}(2\pi r)(l) = \pi r l$.
- **The $\frac{1}{3}$.** A cube cuts into three identical square pyramids. Each one stands on a face of the cube, and all three tips meet at the opposite corner. Each is therefore a third of the cube, and the same fraction holds for every pyramid and every cone: $V = \frac{1}{3}Bh$.
- **Sphere.** Put a sphere inside the smallest cylinder that holds it, of radius r and height $2r$. Archimedes proved the sphere is two thirds of that cylinder. The cylinder holds $\pi r^2(2r) = 2\pi r^3$, so the sphere holds $\frac{4}{3}\pi r^3$. Its surface, $4\pi r^2$, is the area of four **great circles** (a circle round the sphere at its widest).

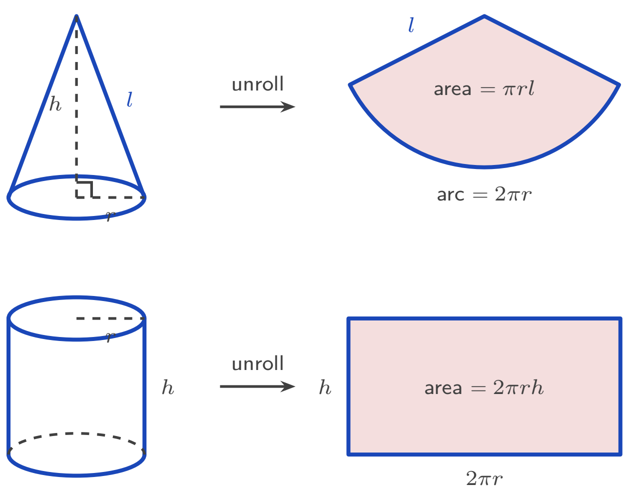

Figure 7.2 Cut the curved surface of a cone along one slant line and it opens into a sector of radius l whose arc is the base circumference. Cut a cylinder the same way and it opens into a rectangle. Each curved-area formula is the area of the flat shape it opens into.

Two mistakes are common. The cone's curved surface uses the slant height l, not the vertical height h; Pythagoras links them by $l = \sqrt{r^2 + h^2}$. And "total surface area" means every face: a solid hemisphere has a curved part $2\pi r^2$ *and* a flat base πr^2, giving $3\pi r^2$, while the same hemisphere as an open bowl has only $2\pi r^2$. For a solid made of two parts, count only the exposed surface: the faces where the parts join are hidden inside and do not count.

Worked through. For a solid made of a cone of base radius 8 and height 6 sitting on top of a hemisphere of radius 8, the exposed surface is the cone's curved part plus the hemisphere's curved part. The two flat circles where they join are hidden inside, so neither counts. The cone has slant height

$$l = \sqrt{8^2 + 6^2} = 10,$$

so its curved area is $\pi r l = 80\pi$. The hemisphere's curved area is $2\pi r^2 = 128\pi$, and the total surface area is

$$80\pi + 128\pi = 208\pi \approx 653.$$

Deciding which faces are exposed is the harder step; the formulas themselves are routine.

7.2.1 Similar solids

Every formula above is a constant times a length squared, for an area, or a length cubed, for a volume. That fixes what happens when a solid is enlarged.

Two solids are **similar** when they have the same shape and all corresponding lengths are in the same ratio. That ratio is the **linear scale factor** k. Multiplying every length by k multiplies $4\pi r^2$ by k^2 and $\frac{4}{3}\pi r^3$ by k^3.

KEY **Scale factors.** If two similar figures have lengths in the ratio k, then

$$\frac{\text{length}_2}{\text{length}_1} = k, \qquad \frac{\text{area}_2}{\text{area}_1} = k^2, \qquad \frac{\text{volume}_2}{\text{volume}_1} = k^3.$$

One ratio of lengths settles every area and every volume.

A cone cut parallel to its base is the case to recognise. The small cone above the cut has the same apex and the same slant, so it is similar to the whole, and one ratio of heights gives its radius, its surface and its volume at once. The piece left below the cut is not similar to either cone, so no scale factor applies to it. Find its volume as the difference between the whole cone and the small cone.

Worked through. A cone has base radius 9 cm and height 12 cm, and a cut parallel to the base is made 4 cm below the apex. To find the radius of the circular cut and the volume of the small cone above it, note that the small cone is similar to the whole cone, and the two heights are known, so

$$k = \frac{4}{12} = \frac{1}{3}.$$

The radius is a length, so it scales by k:

$$r = \tfrac{1}{3}(9) = 3 \text{ cm}.$$

Similar triangles say the same thing directly: the vertical axis and the radius of the cut form a triangle similar to the one made by the axis and the base radius, so $\frac{r}{4} = \frac{9}{12}$.

The whole cone holds

$$V = \tfrac{1}{3}\pi(9)^2(12) = 324\pi \text{ cm}^3,$$

and volume scales by k^3, so the small cone holds

$$\left(\tfrac{1}{3}\right)^3(324\pi) = \tfrac{1}{27}(324\pi) = 12\pi \text{ cm}^3.$$

Computing it directly gives $\frac{1}{3}\pi(3)^2(4) = 12\pi$, which agrees. The piece left below the cut has volume $324\pi - 12\pi = 312\pi$ cm^3, the difference of the two cones.

YOUR TURN 7.2 Volumes and Surface Areas of Solids

Volume and surface area

8. EXACT Find the volume of each solid. Leave your answers in terms of π where appropriate.

(a) sphere of radius 3
(b) cone of radius 3, height 4
(c) pyramid with base area 20, height 9
(d) cylinder of radius 4, height 10

9. EXACT Find the surface area asked for in each case.

(a) the surface area of a sphere of radius 5
(b) the curved surface area of a cone of radius 5, slant height 13
(c) the total surface area of a solid hemisphere of radius 4
(d) the curved surface area of a cylinder of radius 4, height 10
(e) the surface area of an open hemispherical bowl of radius 6 (the outside surface only)

10. EXACT A cone has radius 7 and height 24.

(a) Find its slant height.
(b) Hence find its curved surface area.

Composite solids

11. EXACT A solid is a cylinder of radius 3 and height 10, with a hemisphere of radius 3 fixed to its top face.

(a) Explain which surfaces are exposed.
(b) Find the total surface area, in terms of π.
(c) Find the total volume, in terms of π.

Finding a missing dimension

12. EXACT Work backwards from the given measurement.

(a) A sphere has volume 288π. Find its radius.
(b) A cone of radius 6 has volume 96π. Find its height, then its curved surface area.
(c) A cylinder of height 10 has curved surface area 120π. Find its radius, then its volume.

Similar solids

13. Two similar cones have heights 6 cm and 9 cm. The smaller cone has volume 40 cm^3 and total surface area 24 cm^2.

(a) Find the volume of the larger cone.
(b) Find the total surface area of the larger cone.

14. Two similar jugs hold 250 cm^3 and 2000 cm^3.

(a) Find the ratio of their heights.
(b) The smaller jug has surface area 150 cm^2. Find the surface area of the larger jug.

15. EXACT A cone has base radius 6 cm and height 12 cm. It is cut parallel to its base, 8 cm below the apex, and the small cone above the cut is removed. The piece left below the cut is called a *frustum*.

(a) Find the radius of the circular cut.
(b) Find the volume of the small cone as a fraction of the volume of the whole cone.
(c) Find the volume of the frustum, in terms of π.

Concept check

16. EXACT A cone has base radius 5 and slant height 13. A student writes: "$V = \frac{1}{3}\pi(5^2)(13) = \frac{325\pi}{3}$."

(a) Identify the error.
(b) Find the correct volume, in terms of π.

17. Two jugs are similar, and one is twice as tall as the other. A student says: "*the larger jug is twice as tall, so it holds twice as much.*" Explain the error, and state the correct factor for the capacity.

7.3 The Sine and Cosine Rules

Right-angled triangles are solved with $\sin$, $\cos$ and $\tan$. Name the sides opposite, adjacent and hypotenuse relative to an angle, and $\sin\theta = \frac{o}{h}$, $\cos\theta = \frac{a}{h}$, $\tan\theta = \frac{o}{a}$. Most triangles have no right angle. Two rules extend trigonometry to all of them, and between them they solve any triangle from three known parts, provided at least one of them is a side.

Both rules use one labelling. A side takes the lower-case letter of the angle *opposite* it, so side a faces angle A.

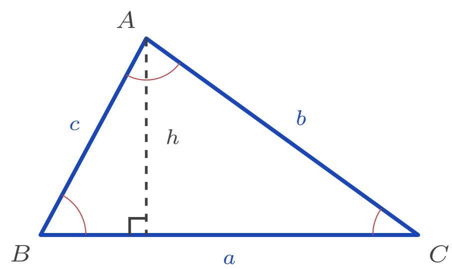

Figure 7.3 Each side is named after the angle opposite it: side a faces angle A, and so on. The dashed altitude h splits the triangle into two right triangles, which is where the sine rule comes from.

7.3.1 The sine rule

Drop a perpendicular of height h from A onto side a, as in the figure above. It cuts the triangle into two right triangles, and in each one h is the side opposite a known angle. On the left, $h = c\sin B$. On the right, $h = b\sin C$. Both expressions measure the same segment, so $c\sin B = b\sin C$, which rearranges to $\frac{b}{\sin B} = \frac{c}{\sin C}$. Dropping a perpendicular from a different corner gives the third ratio.

KEY · IB **Sine rule.** In any triangle,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The sine rule pairs a side with the angle facing it. Use it whenever you know one complete pair, plus one more side or angle. The standard shorthand writes S for a known side and A for a known angle, in the order they occur round the triangle. These are the cases AAS, ASA and SSA.

7.3.2 The ambiguous case

The sine rule needs one extra check when you use it to find an *angle*.

Suppose you know two sides and an angle that is *not* between them. This is the SSA case. The sine rule gives an equation of the form $\sin B = k$. If $k > 1$ there is no solution and no triangle; if $k = 1$ there is one solution, $B = 90^\circ$. If $0 < k < 1$, the equation has two solutions between 0° and 180°: the angle B and its supplement $180^\circ - B$. An angle and its supplement have the same sine, so the equation cannot separate them.

Your calculator returns only the first. Test the second yourself. Add it to the angle you were given: if the total is still below 180°, a positive third angle remains, so a second triangle exists and both answers are correct.

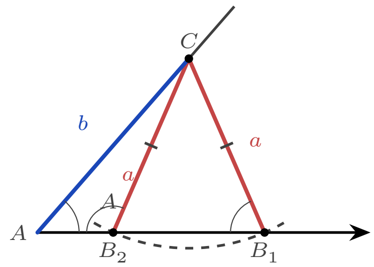

Figure 7.4 Two sides and a non-included angle. Swinging the fixed length a from C meets the base at B_2 and at B_1, so two different triangles share the same a, b and A. The angles at B_2 and B_1 are supplementary, which is why the sine rule returns both.

Worked through. In triangle ABC, $a = 9$, $b = 11$, and $A = 40^\circ$. To find angle B, apply the sine rule: $\dfrac{\sin B}{11} = \dfrac{\sin 40^\circ}{9}$, so

$$\sin B = \frac{11 \sin 40^\circ}{9} \approx 0.786.$$

The calculator gives $B \approx 51.8^\circ$. The supplement is $180^\circ - 51.8^\circ = 128.2^\circ$, so test it: $A + B = 40^\circ + 128.2^\circ = 168.2^\circ$, which is below 180°. A third angle of 11.8° remains, so that triangle exists too.

Both values of B are correct. There are two triangles, and a complete answer reports both.

CAUTION This check is needed only when the sine rule gives an *angle*. If you are finding a *side*, or using the cosine rule, no supplement arises. The cosine rule returns an angle between 0° and 180° directly, because cos is negative for obtuse angles and positive for acute ones, so it distinguishes the two.

7.3.3 The cosine rule

When the known angle sits *between* the two known sides, no side is paired with the angle opposite it, so the sine rule cannot start. The cosine rule covers this case. Read it as Pythagoras' theorem with a correction term for the angle not being 90°.

Coordinates prove it directly. Put the corner C at the origin with side a along the x-axis, so B sits at $(a, 0)$. The point A lies a distance b from the origin, at angle C to the positive x-axis. By the definitions of cosine and sine, its coordinates are $(b \cos C,\ b \sin C)$. The distance formula now measures the third side:

$$c^2 = (a - b\cos C)^2 + (b\sin C)^2 = a^2 - 2ab\cos C + b^2\cos^2 C + b^2\sin^2 C,$$

and $\cos^2 C + \sin^2 C = 1$, so the last two terms add to b^2.

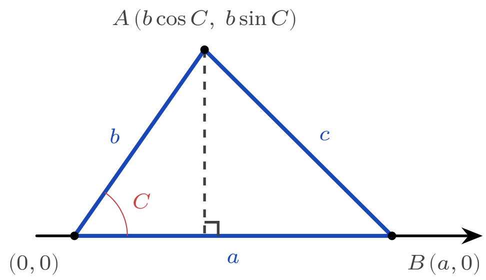

Figure 7.5 Placing C at the origin with a along the axis fixes all three corners. Only A needs trigonometry, and the distance from A to B is then the cosine rule.

KEY · IB **Cosine rule.** In any triangle,

$$c^2 = a^2 + b^2 - 2ab\cos C, \qquad \text{equivalently} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

The letters rotate. Written about a different corner the same rule reads $a^2 = b^2 + c^2 - 2bc\cos A$, and the pattern is the same in each version: the side opposite the named angle stands alone on the left.

When $C = 90^\circ$, $\cos C = 0$ and the rule becomes $c^2 = a^2 + b^2$, which is Pythagoras' theorem. The first form finds a third side from two sides and the angle between them (SAS). The rearranged form finds an angle from all three sides (SSS).

7.3.4 The area of a triangle

The familiar area $\frac{1}{2}$ base $\times$ height becomes a trigonometric formula as soon as the height is written in terms of a side. Take b as the base. The perpendicular height from B down to it is the opposite side of the right triangle at C, so $h = a\sin C$ and

$$\text{Area} = \tfrac{1}{2}\,b\,(a\sin C).$$

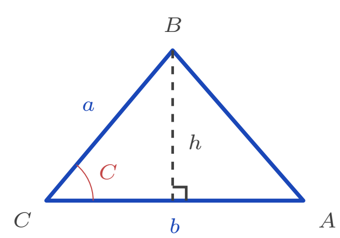

Figure 7.6 The height of the triangle on base b is the opposite side of the right triangle at C, so it is $a \sin C$. Substituting it into $\frac{1}{2} \times$ base $\times$ height gives the formula.

KEY · IB **Area of a triangle.** Area $= \dfrac{1}{2} ab \sin C$, using two sides and the angle between them.

This formula needs exactly what the cosine rule needs, so one sketch usually answers both questions at once.

Worked through. A triangular field has $AB = 90$ m, $AC = 70$ m, and the angle at A is 110°. The known angle lies between the two known sides, so the cosine rule finds the third side BC. Rotate the letters to put A in the named position:

$$BC^2 = 90^2 + 70^2 - 2(90)(70)\cos 110^\circ \approx 17\,309 \implies BC \approx 132 \text{ m}.$$

The area of the field needs the same three pieces of information, so no further measurement is required:

$$\text{Area} = \tfrac{1}{2}(90)(70)\sin 110^\circ \approx 2960 \text{ m}^2.$$

Two sides and the angle between them fix a triangle completely. Once you hold that combination, every other part of the triangle can be found.

What you are given	What to use
Two angles and any side (AAS, ASA)	sine rule
Two sides and an angle opposite one (SSA)	sine rule, plus the ambiguous case
Two sides and the angle between them (SAS)	cosine rule, for the third side
All three sides (SSS)	cosine rule rearranged, for an angle
The same (SAS), when the area is wanted	area $= \frac{1}{2}ab \sin C$

Read the table by counting what you know. Three parts, at least one of them a side, determine the triangle, except that SSA may allow two. The arrangement of the three parts chooses the rule.

YOUR TURN 7.3 The Sine and Cosine Rules

The sine rule

18. Use the sine rule to find the required side or angle.

(a) $a = 7$, $A = 40^\circ$, $B = 60^\circ$: find b
(b) $a = 9$, $A = 35^\circ$, $B = 55^\circ$: find b
(c) $A = 45^\circ$, $B = 60^\circ$, $a = 10$: find C and b
(d) $a = 12$, $b = 10$, $A = 65^\circ$: find B, and explain why there is only one possible value

The cosine rule

19. Use the cosine rule.

(a) $b = 5$, $c = 8$, $A = 60^\circ$: find a
(b) $a = 6$, $b = 9$, $C = 110^\circ$: find c
(c) $a = 6$, $b = 7$, $c = 8$: find angle A
(d) sides 5, 6, 7: find the largest angle

The area of a triangle

20. Find the area of each triangle.

(a) sides 5 and 8, included angle 30°
(b) sides 9 and 6, included angle 125°
(c) sides 7 and 9, included angle 40°
(d) sides 5, 7 and 8: find the angle opposite the side 7 first

21. In triangle PQR, $PQ = 11$ cm, $PR = 7$ cm and $Q\hat{P}R = 52^\circ$.

(a) Find the length QR.
(b) Find the area of the triangle.

The ambiguous case

22. For each triangle ABC, use the sine rule to find $\sin B$. State how many triangles fit the data, and give every possible value of B.

(a) $a = 6$, $b = 10$, $A = 45^\circ$
(b) $a = 4$, $b = 8$, $A = 30^\circ$
(c) $a = 7$, $b = 10$, $A = 35^\circ$

23. In triangle ABC, $a = 7$, $b = 9$ and $A = 45^\circ$.

(a) Find the two possible values of angle B.
(b) Show that both give a valid triangle.
(c) Explain why this situation cannot arise when the given angle is included between the two given sides.
(d) For each triangle, find the length of side c.

Concept check

24. To find a when $b = 5$, $c = 8$ and $A = 60^\circ$, a student writes

$$a^2 = 5^2 + 8^2 - 2(5)(8)\cos 60^\circ = 9\cos 60^\circ = 4.5, \quad \text{so } a \approx 2.12.$$

Identify the error and find the correct value of a.

7.4 Elevation, Depression and Bearings

Real questions do not arrive as labelled triangles. They arrive as heights, distances and directions: how far away is the boat, what heading takes the ship home. Two conventions turn that language into a triangle you can solve.

An **angle of elevation** is measured upward from the horizontal to a line of sight. An **angle of depression** is measured downward from it. The observer's horizontal and the ground are parallel lines. So the angle of depression from a cliff top down to a boat equals the angle of elevation from the boat up to the cliff top. They are alternate angles, and either one can be used in the triangle.

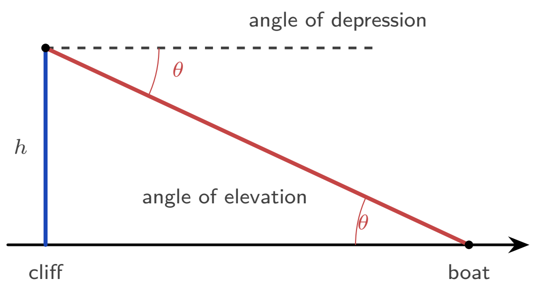

Figure 7.7 The dashed line at the top is the observer's horizontal, and the ground is parallel to it. The line of sight crosses both, so the two marked angles are alternate angles and are equal.

When the distance to the object is unknown, one sighting is not enough to find a height: two unknowns, one equation. A second sighting from a known distance nearer supplies the missing equation. The height is the same from both positions, so writing it twice and equating the two expressions eliminates the distance.

EXAMPLE 7.1

From a point on level ground the angle of elevation of the top of a tower is 25°. From a point 30 m nearer the tower, on the same straight line, the angle of elevation is 40°. Find the height of the tower.

Let h be the height and let d be the distance from the *nearer* point to the foot of the tower. The far point is then $d + 30$ from the foot. Each sighting gives a right triangle with h opposite the angle and the ground distance adjacent to it, so

$$h = d\tan 40^\circ \qquad \text{and} \qquad h = (d+30)\tan 25^\circ.$$

Both measure the same tower, so set them equal and collect the terms in d:

$$d\tan 40^\circ = (d+30)\tan 25^\circ \implies d\,(\tan 40^\circ - \tan 25^\circ) = 30\tan 25^\circ,$$

$$d = \frac{30\tan 25^\circ}{\tan 40^\circ - \tan 25^\circ} \approx \frac{13.99}{0.3728} \approx 37.5 \text{ m}.$$

The height follows from either equation:

$$h = 37.5\tan 40^\circ \approx 31.5 \text{ m}.$$

Check it against the other sighting: the far point is 67.5 m away, and $\arctan\left(\frac{31.5}{67.5}\right) \approx 25^\circ$, as given. Measuring d from the nearer point is what keeps the second distance as $d+30$; taking d from the far point instead makes it $d-30$, and mixing the two gives an impossible answer.

A **bearing** gives a direction as an angle measured clockwise from north. It runs from 000° up to, but not including, 360°, and is always written with three figures, so due east is 090° and not 90°. Three figures keep the reading unambiguous on a chart. Due south is 180°, due west is 270°, and north-west is 315°.

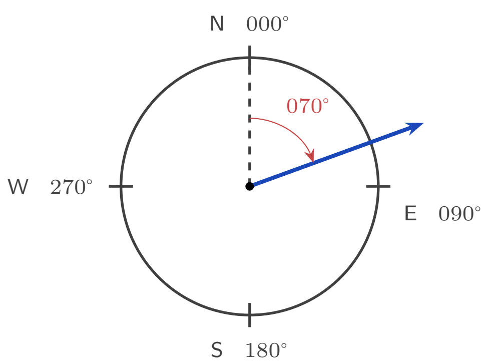

Figure 7.8 A bearing is measured clockwise from north, always with three figures. The arrow here is on a bearing of 070°.

The **back bearing** is the direction of the return journey, and it differs from the outward bearing by 180°. Add 180° when the bearing is below 180°, and subtract 180° when it is 180° or more. Either way the answer stays between 000° and 360°.

EXAMPLE 7.2

A ship sails 60 km from port P on a bearing of 050° to a point Q, then 80 km on a bearing of 140° to R. Find the distance PR.

The two bearings differ by $140^\circ - 50^\circ = 90^\circ$, so the ship turns through a right angle at Q: the angle PQR is 90°. Then PR is a hypotenuse, and Pythagoras gives

$$PR = \sqrt{60^2 + 80^2} = \sqrt{10000} = 100 \text{ km}.$$

Sketching the bearings from a north line at P and at Q is what shows the right angle. The arithmetic then follows from the picture.

Most journeys do not turn so conveniently. When the corner angle is not 90°, read the two distances and that angle off the same sketch. Find the missing distance with the cosine rule, then find the direction back with the sine rule. Many bearings questions have this shape, and the steps below handle them in the same order.

KEY **Solving a bearings problem.**

1. Sketch the journey, and draw a north line at **every** point where a bearing is given.
2. Mark each bearing as an angle clockwise from its own north line.
3. Find the interior angle of the triangle at the corner: the difference between the back bearing of the first leg and the bearing of the second leg. If that difference is more than 180°, subtract it from 360°.
4. Use the cosine rule for the missing distance, then the sine rule for the missing angle.
5. Turn that angle back into a bearing: add it to the bearing of the first leg if the route turns right (clockwise) at the corner, and subtract it if the route turns left. The sketch shows which.

Do not skip step 1: without a north line drawn at the corner, step 3 is easy to get wrong.

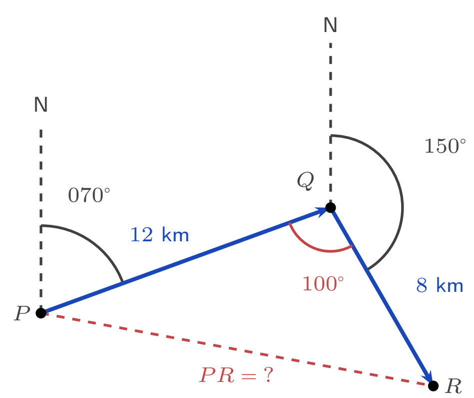

Figure 7.9 The two bearings are measured clockwise from a north line at P and at Q. Together they fix the interior angle $P\hat{Q}R = 100^\circ$, which is what the cosine rule needs.

Worked through. For the journey in the figure, a hiker walks 12 km from P on a bearing of 070° to Q, then 8 km on a bearing of 150° to R. To find the distance PR and the bearing of R from P, first construct the angle at the corner. The bearing back from Q to P is $070^\circ + 180^\circ = 250^\circ$, and the onward bearing is 150°, so the interior angle is $P\hat{Q}R = 250^\circ - 150^\circ = 100^\circ$. Subtracting the second bearing from the back bearing is the standard method, and the sketch agrees: the angle at the corner is obtuse. There is no right angle, so the cosine rule finds the closing side:

$$PR^2 = 12^2 + 8^2 - 2(12)(8)\cos 100^\circ \approx 241.3 \implies PR \approx 15.5 \text{ km}.$$

For the bearing of R from P, find the angle $Q\hat{P}R$ by the sine rule:

$$\frac{\sin(Q\hat{P}R)}{8} = \frac{\sin 100^\circ}{15.54} \implies \sin(Q\hat{P}R) \approx 0.507 \implies Q\hat{P}R \approx 30.5^\circ.$$

The route turns right at Q, so add. The bearing of R from P is $070^\circ + 30.47^\circ \approx 100.47^\circ$, which is 100° to the nearest degree. Every step came from the labelled diagram: the bearings give the corner angle, the cosine rule the closing side, and the sine rule the final direction.

YOUR TURN 7.4 Elevation, Depression and Bearings

Elevation and depression

25. Find the angle of elevation in each case.

(a) a 50 m tower, from 120 m away on level ground

(b) a 45 m tower, from 90 m away on level ground

26. In each case, find the horizontal distance asked for.

(a) From the top of an 80 m cliff, the angle of depression to a boat is 25°. Find the distance from the foot of the cliff to the boat.

(b) From the top of a 30 m building, the angle of depression to a car is 40°. Find the distance from the foot of the building to the car.

(c) From the top of a 60 m cliff, the angles of depression to two buoys are 20° and 35°. The buoys and the foot of the cliff lie on one straight line, with both buoys on the same side of the cliff. Find the distance between the buoys.

Two sightings

27. From a point A on level ground, the angle of elevation of the top of a building is 30°. From a point B, 50 m nearer the building on the same straight line, the angle of elevation is 45°. Find the height of the building. *(Hint: let d be the distance from B to the foot of the building, and write the height twice)*

28. From a boat, the angle of elevation of the top of a lighthouse is 12°. The boat sails 200 m directly towards the lighthouse, and the angle of elevation is now 20°.

(a) Find the height of the lighthouse.
(b) Find the distance from the boat to the foot of the lighthouse at the second sighting.

Bearings

29. Answer each question about bearings.

(a) Write the bearing of due west as a three-figure bearing.
(b) A ship travels on a bearing of 120°. Find its back bearing.
(c) B is on a bearing of 065° from A. Find the bearing of A from B.
(d) Find the back bearing of 200°, and explain why you subtract here rather than add.

Journeys with two legs

30. A ship sails 15 km from P on a bearing of 040° to Q, then 20 km on a bearing of 110° to R.

(a) Show that angle $P\hat{Q}R = 110^\circ$.
(b) Find the distance PR.
(c) Find the bearing of R from P.

31. A hiker walks 25 km from P on a bearing of 080° to Q, then 18 km on a bearing of 330° to R.

(a) Find the angle $P\hat{Q}R$.
(b) Find the distance PR.
(c) Find the bearing of R from P.
(d) Find the bearing of P from R.

Concept check

32. From the top of a 50 m cliff, the angle of depression to a boat is 20°. A student places the 20° between the cliff face and the line of sight, and writes
$d = 50 \tan 20^\circ \approx 18.2$ m.

(a) Identify the error.
(b) Find the correct distance from the foot of the cliff to the boat.

7.5 Radians, Arc Length, and Sector Area

The degree is a human invention. The Babylonians cut the circle into 360 parts, and we inherited the choice. Nothing about a circle requires 360. The circle does suggest a unit of its own: take the radius, bend it round the circumference, and measure the angle it makes at the centre. That angle is one **radian**. Because the circumference is $2\pi r$, exactly 2π radii fit around it, so a full turn is 2π radians and one radian is about $57.3°$.

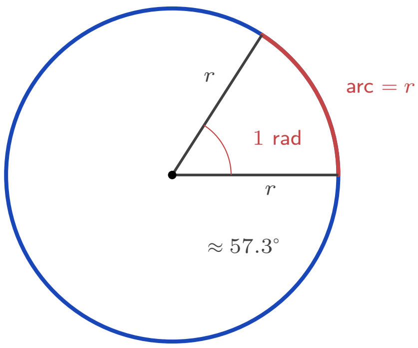

Figure 7.10 One radian is the angle at the centre for which the arc is exactly as long as the radius. A full turn takes 2π of them, because the circumference is 2π radii long.

KEY 2π radians $= 360°$, so π radians $= 180°$. Convert by multiplying: degrees to radians $\times \dfrac{\pi}{180}$; radians to degrees $\times \dfrac{180}{\pi}$.

The angles that appear most often should be known without conversion.

Degrees	$30°$	$45°$	$60°$	$90°$	$180°$	$360°$
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	2π

The reason for the new unit appears at once: arc length and sector area become simple fractions of the whole circle. An arc is the same fraction of the circumference as its angle θ is of a full turn, so $\dfrac{l}{2\pi r} = \dfrac{\theta}{2\pi}$, and the two 2π's cancel.

KEY · IB For a sector of radius r and angle θ **in radians**,

$$\text{arc length } l = r\theta, \qquad \text{sector area } A = \tfrac{1}{2}r^2\theta.$$

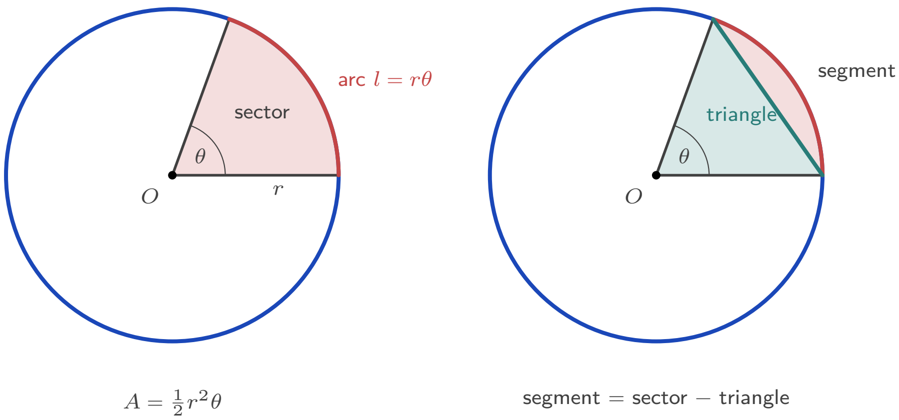

Figure 7.11 **Left:** the *sector* is the pie slice between two radii and the arc. **Right:** the chord cuts that same sector into a triangle and a *segment*, so subtracting the triangle's area from the sector's leaves the segment.

The area works the same way. The sector is the fraction $\dfrac{\theta}{2\pi}$ of the whole disc πr^2, so $A = \dfrac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2}r^2\theta$. A **segment** is the region between a chord and its arc. It is a sector with the triangle removed, and that triangle has two sides r with the angle θ between them, so §7.3 gives its area as $\frac{1}{2}r^2 \sin\theta$:

$$\text{segment} = \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta = \tfrac{1}{2}r^2(\theta - \sin\theta).$$

CAUTION Every formula in this section needs θ in **radians**. Substituting an angle in degrees into $l = r\theta$ or $A = \frac{1}{2}r^2\theta$ gives an answer that is wrong by a factor of $\frac{180}{\pi}$ (for the segment formula the error is not a fixed factor), and the working looks correct, so the error is easy to miss. Check the mode on your calculator before you start.

Worked through. For a sector of radius 10 cm and angle $\frac{2\pi}{5}$ radians, the formulas give the arc length and area directly:

$$l = r\theta = 10 \cdot \frac{2\pi}{5} = 4\pi \approx 12.6 \text{ cm}, \qquad A = \tfrac{1}{2}r^2\theta = \tfrac{1}{2}(100)\frac{2\pi}{5} = 20\pi \approx 62.8 \text{ cm}^2.$$

For the segment cut off by the chord, subtract the triangle from the sector:

$$\tfrac{1}{2}r^2(\theta - \sin\theta) = 50\left(\frac{2\pi}{5} - \sin\frac{2\pi}{5}\right) \approx 50(1.2566 - 0.9511) \approx 15.3 \text{ cm}^2.$$

Each formula took the angle in radians, exactly as given. The segment is far smaller than the sector, because most of the sector is triangle.

A sector's boundary is not the arc alone. Walk round the edge and you travel out along one radius, round the arc, and back along the other radius. The perimeter of a sector is therefore the arc plus two radii, and it has to be assembled from $l = r\theta$ rather than looked up, because the formula booklet does not print it.

KEY **Perimeter of a sector** of radius r and angle θ in radians:

$$P = 2r + r\theta = r(2 + \theta).$$

Two straight radii and one curved arc. This one is not in the formula booklet.

EXAMPLE 7.3

A sector has radius 7 cm and angle 1.5 radians. Find its perimeter.
The arc is

$$l = r\theta = 7(1.5) = 10.5 \text{ cm},$$

and the two radii add 14 cm, so

$$P = 2(7) + 10.5 = 24.5 \text{ cm}.$$

A quick sanity check: the two radii alone measure 14 cm, and an angle of 1.5 radians is just under a quarter turn, so the arc should be 1.5 radii long, which 10.5 cm is.

Given the perimeter and the area together, neither formula can be used on its own, because each contains both r and θ. Treat them as simultaneous equations, and eliminate θ. The resulting quadratic usually gives two values of r. Find θ for each, and reject any value with $\theta > 2\pi$, since no sector has an angle of more than a full turn.

EXAMPLE 7.4

A sector has perimeter 36 cm and area 45 cm^2. Find its radius and its angle.
Write down both conditions:

$$2r + r\theta = 36, \qquad \tfrac{1}{2}r^2\theta = 45.$$

The second gives $r^2\theta = 90$, so $\theta = \dfrac{90}{r^2}$. Substituting that into the first,

$$2r + r \cdot \frac{90}{r^2} = 36 \implies 2r + \frac{90}{r} = 36.$$

Multiply through by r and collect:

$$2r^2 - 36r + 90 = 0 \implies r^2 - 18r + 45 = 0 \implies (r-3)(r-15) = 0,$$

so $r = 3$ or $r = 15$. Test each against $\theta = \dfrac{90}{r^2}$. With $r = 3$ the angle is $\theta = 10$ radians, which is more than a full turn of $2\pi \approx 6.28$, so no such sector exists. With $r = 15$ the angle is $\theta = 0.4$ radians.
So $r = 15$ cm and $\theta = 0.4$ radians. Check: the arc is $15(0.4) = 6$ cm, giving a perimeter of $30 + 6 = 36$ cm, and the area is $\frac{1}{2}(225)(0.4) = 45$ cm^2. ✓ The quadratic will usually offer two radii, and rejecting the one whose angle exceeds 2π is part of the answer, not an afterthought.

EXAMPLE 7.5

The opener measured the Earth from a hilltop. There is an older method, and it uses this section's formula. Eratosthenes measured the sun's rays at 7.2° to the vertical in Alexandria, while in Syene they were vertical. He took the arc between the towns to be 5000 stadia. Find the Earth's radius and circumference in stadia.

Convert the angle first:

$$7.2^\circ \times \frac{\pi}{180} = \frac{\pi}{25} \text{ radians.}$$

The two towns and the Earth's centre make a sector whose arc is the distance between them, so $l = r\theta$ can be used in reverse:

$$5000 = r \cdot \frac{\pi}{25} \implies r = \frac{125\,000}{\pi} \approx 39\,800 \text{ stadia,}$$

and the circumference is $2\pi r = 250\,000$ stadia. Since 7.2° is $\frac{1}{50}$ of a full turn, this is Eratosthenes' own argument: the circumference is 50 times the arc. Taking a stadion as roughly 158 m gives a circumference near $39\,500$ km, within a couple of percent of the true $40\,075$ km.

YOUR TURN 7.5 Radians, Arc Length, and Sector Area

Degrees and radians

33. EXACT Convert to radians, in terms of π.

(a) 60°
(b) 135°
(c) 210°
(d) 90°

34. Convert to degrees.

(a) $\dfrac{3\pi}{4}$
(b) $\dfrac{2\pi}{3}$
(c) 1 radian, to three significant figures

Arc length and sector area

35. Find the arc length and the area of each sector.

(a) radius 8, angle 1.5 radians
(b) radius 6, angle $\dfrac{\pi}{3}$ radians
(c) radius 9, angle 2.4 radians
(d) radius 4, angle 90°

36. In each case the angle is unknown. Find it from the information given.

(a) A sector has radius 10 and arc length 25. Find its angle in radians.
(b) A sector has radius 6 and area 54. Find its angle in radians, then find its arc length.

Segments and perimeters

37. A chord subtends an angle of 2 radians at the centre of a circle of radius 8.

(a) Find the area of the sector.

(b) Find the area of the triangle formed by the chord and the two radii.

(c) Find the area of the minor segment.

(d) Find the perimeter of the sector.

Perimeter and area together

38. In each case, find the radius and the angle of every sector that fits the information.

(a) The perimeter is 40 cm and the area is 64 cm^2.

(b) The perimeter is 20 cm and the area is 24 cm^2.

Concept check

39. EXACT A student finds the arc length of a sector of radius 6 and angle 60° as $s = r\theta = 6 \times 60 = 360$. Identify the error and find the correct arc length.

Chapter 7 · Shapes & Measures

Reflections

TOK The formulas $l = r\theta$ and $A = \frac{1}{2}r^2\theta$ are this simple only when θ is in radians, and later you will meet another case: the derivative of $\sin x$ is exactly $\cos x$, again only in radians. The degree was chosen by people, the Babylonians. The radian is fixed by the circle: it is the angle whose arc equals its radius. Does a unit that the circle chooses make the mathematics truer, or only simpler?

IM Trigonometry was built by many people over many centuries. The Greek astronomer Hipparchus made the first table of chords around 150 BCE. In India, Aryabhata replaced the chord with the half-chord and called it the *jyā*, which is our sine. The word passed into Arabic, was misread there as *jaib*, meaning "bay", and was translated into Latin as *sinus*. Islamic astronomers, al-Bīrūnī among them, extended the sine and cosine rules, partly to find the direction of Mecca from any city.

RW Fixing an unknown position from measured angles is called **triangulation**. Surveyors still use it, and once carried chains of triangles across whole countries. Astronomers use *parallax*: as the Earth moves round the Sun, a near star seems to shift against the far ones, and that small angle gives its distance. Your phone does the same job with distances instead of angles. A GPS receiver times the signal from each satellite, turns each time into a distance, and finds the one point lying at all of those distances at once: three satellites fix a point on the Earth's surface, and a fourth corrects the receiver's clock. This is **trilateration**, the sphere equation of §7.1 solved several times over.

EXT The ambiguous case is really a statement about congruence. Give a triangle by two angles and a side (ASA or AAS), by three sides (SSS), or by two sides and the angle between them (SAS), and only one triangle fits. Those are the congruence rules. Two sides and an angle that is *not* between them (SSA) is not one of them, so it can leave the triangle undecided. The sine rule's two answers are the two triangles that SSA allows.

End-of-Chapter Problems
Chapter 7: Shapes & Measures

1.

(a) Convert $\frac{5\pi}{6}$ radians to degrees. [1]

(b) Convert $\frac{7\pi}{4}$ radians to degrees. [1]

(c) A sector has radius 9 cm and angle $\frac{5\pi}{6}$. Find its arc length, giving your answer exactly. [2]

2. A pendulum of length 0.8 m swings through an angle of 0.5 radians. Find the length of the arc traced by the bob. [2]

3. A cuboid has dimensions $6 \times 8 \times 10$.

(a) Find the length of the space diagonal. [2]

(b) Find the angle between the space diagonal and the 6×8 face. [3]

4. GDC Consider the points $A(2, 0, 0)$, $B(0, 4, 0)$ and $C(0, 0, 4)$.

(a) Find the lengths AB, AC and BC. [3]

(b) Find angle BAC. [3]

5. EXACT A cone has radius 5 and height 12.

(a) Find the slant height. [1]

(b) Find the volume. [2]

(c) Find the total surface area. [2]

6. EXACT A solid consists of a cylinder of radius 3 and height 10 with a hemisphere of radius 3 on top. Find its total surface area. [5]

7. GDC In triangle ABC, $a = 8$, $b = 11$, $c = 15$.

(a) Find the largest angle. [3]

(b) Find the area of the triangle. [2]

8. GDC In triangle PQR, $p = 10$, $q = 7$ and $R = 95°$.

(a) Find r. [2]

(b) Find the area of the triangle. [2]

9. GDC A triangle has two sides of length 9 and 12 with an included angle of $40°$.

(a) Find the length of the third side. [2]
(b) Find the area of the triangle. [2]

10. GDC A boat sails 30 km from P on a bearing of $100°$ to Q, then 20 km on a bearing of $040°$ to R.

(a) Show that angle $PQR = 120°$. [3]
(b) Find the distance PR. [3]
(c) Find the bearing of R from P. [4]

11. A sector has radius 12 cm and angle 1.2 radians.

(a) Find the arc length. [1]
(b) Find the area of the sector. [2]
(c) Find the perimeter of the sector. [2]

12. A sector has area 30 cm^2 and radius 6 cm.

(a) Find the angle of the sector in radians. [2]
(b) Find the arc length. [2]

13. GDC A chord subtends an angle of 1.5 radians at the centre of a circle of radius 10 cm.

(a) Find the area of the sector. [2]
(b) Find the area of the triangle formed by the chord and the two radii. [2]
(c) Find the area of the minor segment. [2]

14. GDC In triangle ABC, $A = 50°$, $a = 6$ and $b = 7$.

(a) Find the two possible values of angle B. [3]
(b) Find the two possible areas of the triangle. [3]

15. EXACT A regular hexagon is inscribed in a circle of radius 8. Find its area. [4]

16. EXACT A cone has volume 100π and radius 5.

(a) Find the height. [2]
(b) Find the slant height. [2]
(c) Find the curved surface area. [1]

17. GDC From the top of a vertical cliff 80 m high, the angles of depression of two boats are $32°$ and $18°$. The boats and the foot of the cliff lie on one straight line, with both boats on the same side of the cliff. Find the distance between the two boats. [4]

18. A square-based pyramid has base side 10 and height 12.

(a) Find the volume. [2]
(b) Find the slant height of a triangular face. [2]
(c) Find the total surface area. [2]

19. GDC The area of a triangle is 24, with $a = 8$ and $b = 10$. Find the two possible values of the included angle C. [3]

20. GDC Two ships leave a port at the same time. One sails on a bearing of 040° at 12 km/h, the other on a bearing of 100° at 16 km/h. Find the distance between them after 2 hours. [4]

21. EXACT The points $A(2, -1, 3)$ and $B(4, 4, 7)$ are given, and O is the origin.

(a) Find AB. [2]
(b) Find the midpoint of $[AB]$. [1]
(c) Show that $OB = 9$. [1]

22. A windscreen wiper arm OA of length 50 cm turns about O through an angle of 1.8 radians. The rubber blade covers the part of the arm between 20 cm and 50 cm from O.

(a) Find the length of the arc traced by the tip A. [2]
(b) Find the area of glass wiped by the blade. [4]
(c) Find the perimeter of the wiped region. [3]

23. GDC Two observers stand 200 m apart on level ground, with a balloon in the vertical plane between them. From one the angle of elevation to the balloon is 40°; from the other it is 55°. Find the height of the balloon. [6]

24. GDC In triangle ABC, $AB = 12$, $BC = 9$ and angle $BAC = 35^\circ$. Find the two possible values of angle BCA. [4]

25. GDC A sector of a circle of radius 12 and angle 1.5 radians is rolled into a cone, so that the radius becomes the slant height and the arc becomes the base circumference. Find the radius and height of the cone. [6]

26. A sector of a circle has perimeter 26 cm and area 30 cm^2. Find the radius r and the angle θ of the sector. [6]

27. GDC A chord of a circle of radius 10 has length 12. Find the area of the minor segment it cuts off. [6]

28. GDC The area of a triangle is 84, and two of its sides have lengths 13 and 14. Find both possible lengths of the third side. [6]

29. GDC A yacht leaves harbour H and sails 25 km on a bearing of 048° to a buoy A. It then sails 18 km on a bearing of 130° to a second buoy B.

(a) Show that angle $HAB = 98^\circ$. [3]
(b) Find the distance HB. [4]
(c) Find the bearing of B from H. [4]
(d) Find the area of triangle HAB. [3]
(e) The yacht returns directly from B to H. Find the total distance it has sailed. [2]

30. GDC A rectangular box $ABCDEFGH$ has a horizontal base $ABCD$ with $AB = 8$ cm and $BC = 6$ cm, and vertical edges of length 5 cm, so that G is the vertex above C.

(a) Find the length of the base diagonal AC. [2]
(b) Find the exact length of the space diagonal AG. [3]
(c) Find the angle that AG makes with the base $ABCD$. [3]
(d) Find the angle $G\hat{A}B$ between the diagonal AG and the edge AB. [4]
(e) Find the total surface area of the box. [3]

31. EXACT A container is a cone of base radius 6 cm and height 15 cm, held with its point downwards.

(a) Find the volume of the cone when full, in terms of π. [2]
(b) Water is poured in to a depth of 10 cm. Using similar triangles, find the radius of the circular water surface. [3]
(c) Find the volume of water, in terms of π. [3]
(d) Find the fraction of the cone's volume that is filled, and explain why it is not $\frac{2}{3}$ even though the water reaches $\frac{2}{3}$ of the height. [4]
(e) Find the depth of water when the cone is exactly half full, giving your answer to three significant figures. [3]

Challenge Questions

32. GDC **How many triangles fit the data?** In triangle ABC you are given the side b, the angle A, and the side a opposite A. This is the SSA case. This investigation finds exactly how many triangles fit, for every possible value of a, first from a diagram and then from a quadratic equation.

In parts (a) to (e), take A to be acute, and let h be the distance from C to the line AB.

(a) Show that $h = b \sin A$. [2]

(b) Take $b = 10$ and $A = 40^\circ$. Find the number of triangles when $a = 5$, when $a = 8$, and when $a = 12$. [3]

(c) Explain why no triangle exists when $a < b \sin A$. *(Hint: what is the shortest distance from C to the line AB?)* [2]

(d) Explain why exactly one triangle exists when $a = b \sin A$, and state what is special about it. [2]

(e) Show that two triangles exist exactly when $b \sin A < a < b$, and that only one exists when $a > b$. Explain what happens at $a = b$. [4]

(f) Now let the side $c = AB$ be the unknown. Use the cosine rule to show that

$$c^2 - (2b \cos A)\, c + (b^2 - a^2) = 0,$$

and use this equation to find both values of c when $b = 10$, $A = 40^\circ$ and $a = 8$. [4]

(g) Each positive root c of this quadratic gives one triangle. Use its discriminant, and the sum and product of its roots, to prove the counts of parts (c) to (e) again, including the case $a = b$. Then use the same method to find how many triangles exist when A is obtuse, for $a \leq b$ and for $a > b$. [5]

(h) Take $b = 10$ and $A = 40^\circ$ again. Find both values of c when $a = 6.44$ and when $a = 6.5$. Comment on what this means for a surveyor whose measurement of a carries a small error when a is close to $b \sin A$, and suggest how this could be investigated further. [3]

33. GDC **The best cone from a sector.** A sector of radius R and angle θ radians is cut from a sheet of paper. The two straight edges are joined to make a cone. This investigation finds the angle that gives the largest cone.

Write $t = \dfrac{\theta}{2\pi}$, so that $0 < t < 1$.

(a) The arc of the sector becomes the circumference of the base. Show that the base radius is $r = Rt$. [2]

(b) Show that the height of the cone is $h = R\sqrt{1 - t^2}$. [3]

(c) Hence show that the volume is $V = \frac{1}{3}\pi R^3 t^2 \sqrt{1 - t^2}$. [2]

(d) Taking $R = 1$, use technology to find the value of t that makes V largest, and hence find θ to the nearest degree. [3]

(e) Let $u = t^2$. Show that

$$\frac{4}{27} - u^2(1 - u) = \left(u - \frac{2}{3}\right)^2 \left(u + \frac{1}{3}\right),$$

and hence verify that V is largest when $t^2 = \frac{2}{3}$. Find the exact values of $\dfrac{r}{R}$ and $\dfrac{h}{r}$ at this maximum. *(Hint: V is largest where V^2 is largest, and V^2 is a multiple of $u^2(1 - u)$)* [5]

(f) Now the whole disc of radius R is cut into two sectors, a fraction t and a fraction $1-t$ of the disc, and each is made into a cone. Write down the total volume of the two cones as a function of t, and use technology to find the value of t that makes it largest. Compare the result with part (d), and suggest how this could be investigated further. [4]

34. How reliable was al-Bīrūnī's measurement? The opener described his method. From a hill of height h he measured the angle of dip θ to the horizon, and the radius of the Earth follows from one right-angled triangle. This investigation asks how much trust the answer deserves.

(a) The line of sight touches the Earth, so it meets the radius there at a right angle. Show that $\cos\theta = \dfrac{R}{R+h}$, and hence that $R = \dfrac{h\cos\theta}{1-\cos\theta}$. [3]

(b) Al-Bīrūnī recorded $h = 652$ cubits and $\theta = 34'$. Find R in cubits. *(Hint: $34' = \frac{34}{60}$ of a degree)* [3]

(c) Repeat the calculation for $\theta = 33'$ and for $\theta = 35'$. Find the percentage change in R in each case. [4]

(d) For small θ the approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ holds, with θ in radians. Use it to show that $R \approx \dfrac{2h}{\theta^2}$. [3]

(e) Hence explain why an error of about 3% in θ produces an error of about 6% in R. Comment on what this means for how confidently his result can be quoted. [4]

(f) Suppose his instrument could read any angle to within about δ radians, however large the angle. Use part (d) to show that the percentage error in R is then roughly proportional to $\dfrac{1}{\sqrt{h}}$, and state what this suggests about the choice of hill. Suggest one further effect that an extended investigation of the method should include. [3]

35. GDC **A line's angles with three perpendicular faces.** A line leaves the corner O of a cuboid. It makes some angle with each of the three faces that meet at O. This investigation finds the rule that links the three angles.

The cuboid has edges of lengths p, q and r along the x-, y- and z-axes, with O at the origin, and $G(p,q,r)$ is the opposite corner. Let α, β and γ be the angles that the diagonal OG makes with the faces $x=0$, $y=0$ and $z=0$ respectively.

(a) Take $p=2$, $q=3$ and $r=6$. Find the length OG, and find α, β and γ in degrees, correct to one decimal place. [4]

(b) Find $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ for the cuboid in part (a) and for the cuboid with $p=1$, $q=2$, $r=2$. State a conjecture. [2]

(c) Prove your conjecture for every cuboid. *(Hint: the perpendicular from G to the face $x=0$ has length p)* [3]

(d) Explain why any line through O that enters the region $x>0$, $y>0$, $z>0$ is the diagonal of some cuboid with a corner at O. State what the rule becomes for a line that lies in the face $z=0$, and what it then says about $\alpha+\beta$. [3]

(e) Show that no line can make an angle of $30°$ with each of the three faces. Find the exact value of $\sin \alpha$ when the three angles are equal, and name the cuboid whose diagonal does this. [3]

(f) Find $\alpha + \beta + \gamma$ for the three lines met in parts (a), (b) and (e), and for a line in the face $z = 0$. Conjecture the least and greatest possible values of $\alpha + \beta + \gamma$, and suggest how this investigation could be extended. [3]