FUNCTIONS

CHAPTER 6

6

Polynomials

SYLLABUS 1.11 2.12 HL ONLY

A car's speed is recorded once every second of a test run. Nobody wrote down a formula for it, yet a polynomial drawn through those readings predicts the speed between them.

Many quantities change smoothly rather than in sudden jumps. Temperature, water level and the position of a moving object are examples, and polynomials are among the simplest functions that change in the same way. The degree matches the situation: distance travelled at a constant speed is linear, a thrown ball follows a parabola, and the volume of a sphere grows with the cube of its radius. Over a short enough interval almost any smooth curve can be matched closely by a polynomial, so measurements are often fitted with one when no exact formula is known. See Ch. 16.

That smoothness comes from the way a polynomial is built. It uses one variable and a few constants, combined by addition and multiplication only. There is no division by the variable, no square root and no trigonometry. A polynomial is therefore defined for every real number, and its graph is a smooth, unbroken curve with no gaps, no asymptotes and no corners.

Describing a polynomial is easy. Solving one exactly can be hard. A general formula for the roots, built from the coefficients using arithmetic and roots such as $\sqrt{}$ and $\sqrt[3]{}$, exists for degree 2, degree 3 and degree 4. Niels Abel proved in 1824 that for degree 5 and above no general formula of this kind exists, although some particular equations of higher degree can still be solved exactly. By the end of this chapter you will be able to read a polynomial's graph from its factorised form, divide one polynomial by another, test whether a value is a root and factorise once you have found one, and give the sum and the product of the roots from the coefficients alone.

One idea runs through most of the chapter: a polynomial's coefficients and its roots describe the same object, and most techniques here move between the two. The chapter closes with **partial fractions**, which splits a fraction by factorising its denominator and giving each factor a numerator of its own. That is why it belongs here rather than with the rational functions of §5.4, and it makes many rational functions straightforward to integrate later.

See Ch. 15.

6.1 Polynomial Functions and their Graphs

Much of the graph of a polynomial can be described before a single point is plotted, because the degree and the coefficients already carry that information. Reading those features from the equation is faster than building a table of values, and they are what a sketch has to show.

DEF A **polynomial function** of **degree** n has the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0,$$

where the coefficients are constants and n is a non-negative integer.

6.1.1 Degree, Coefficients and End Behaviour

The graph features in this subsection are read from two things: the degree, and the sign of the leading coefficient.

Polynomials are named by their degree. Degree 1 is **linear**, degree 2 is **quadratic**, degree 3 is **cubic**, degree 4 is **quartic**, and degree 5 is **quintic**. The degree also limits the shape: how many times the curve can meet the x-axis, and how often it can change direction.

The constants $a_n, a_{n-1}, \dots, a_1, a_0$ are the **coefficients** of the powers of x. The coefficient of the highest power is a_n, called the **leading coefficient**, and the term $a_n x^n$ that contains it is the **leading term**.

For a factorised polynomial, neither the degree nor the leading coefficient needs the brackets multiplied out. The degree is the sum of the degrees of the factors, which for linear factors is simply how many there are. The leading coefficient is the product of the leading coefficients of the factors.

A coefficient may take any value, with one restriction: the leading coefficient cannot be zero. If a_n were zero, the term $a_n x^n$ would be zero, and what remained would be a polynomial of degree $n - 1$ instead.

The sign of the leading coefficient determines which way the ends of the curve point. It says nothing about the sign of the function elsewhere: $y = x^2 - 4$ has a positive leading coefficient, yet y is negative for $-2 < x < 2$.

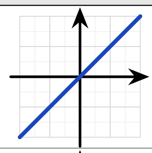
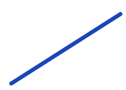
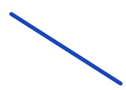
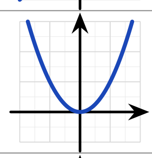
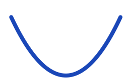
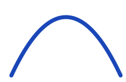
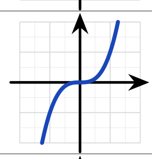
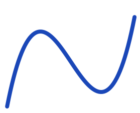
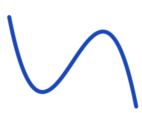
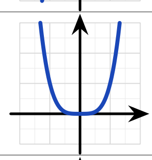
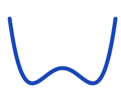
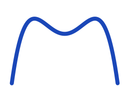
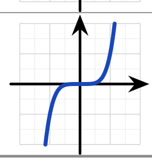
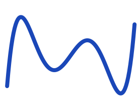
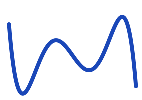
Three features can be read from the equation.

The **end behaviour**, meaning the direction of the curve as $x \to \pm\infty$, is determined entirely by the leading term. If n is odd, the two ends point in opposite directions. If n is even, they point the same way, and the sign of a_n determines whether that direction is upward or downward.

The **zeros** of f are the values of x for which $f(x) = 0$; they are also called the **roots** of the equation $f(x) = 0$, and this chapter uses both words. The real zeros are the x-intercepts of the graph, and the number of distinct real zeros can never exceed the degree. A cubic meets the axis at most three times, a quartic at most four.

A **turning point** is a place where the curve stops rising and begins to fall, or stops falling and begins to rise. A polynomial of degree n has at most $n-1$ turning points. The possible numbers then decrease in steps of two: $n-1$, $n-3$, and so on. The number changes by two rather than by one because the degree fixes the direction of the two ends. Removing a maximum without also removing a minimum would reverse one end, so the number of turning points is always odd when n is even, and even when n is odd.

The table sets out all three features for the degrees you meet most often.

n	$y = x^n$	**Shape when** $a_n > 0$	**Shape when** $a_n < 0$	**Times it meets the** x**-axis**	**Turning points**
1				1	0
2				0, 1 or 2	1
3				1, 2 or 3	0 or 2
4				0, 1, 2, 3 or 4	1 or 3
5				1, 2, 3, 4 or 5	0, 2 or 4

Worked through. To state the degree, the leading coefficient and the two end behaviours of (a) $p(x) = 5 - 4x + 6x^3 - x^2$ and (b) $q(x) = (1-2x)(x+3)^2(x-4)$, work from the leading term in each case.

(a) The terms are not written in order of power, so find the highest power rather than reading the first term. It is $6x^3$, so the degree is 3 and the leading coefficient is 6. An odd degree points the two ends in opposite directions, and a positive leading coefficient sends the right-hand end upward:

$$p(x) \to +\infty \text{ as } x \to +\infty, \qquad p(x) \to -\infty \text{ as } x \to -\infty.$$

(b) Nothing needs to be multiplied out. Count the linear factors: $(1-2x)$, two copies of $(x+3)$, and $(x-4)$, so the degree is 4. The leading coefficient is the product of the four leading coefficients,

$$(-2) \times 1 \times 1 \times 1 = -2.$$

An even degree points both ends the same way, and the negative leading coefficient sends them down: $q(x) \to -\infty$ as $x \to \pm\infty$.

The answer to (b) depends on the -2, which comes from the factor written backwards as $(1-2x)$. Reading the coefficient of x in that bracket as 2 and losing the minus sign gives a leading coefficient of $+2$, which reverses both ends.

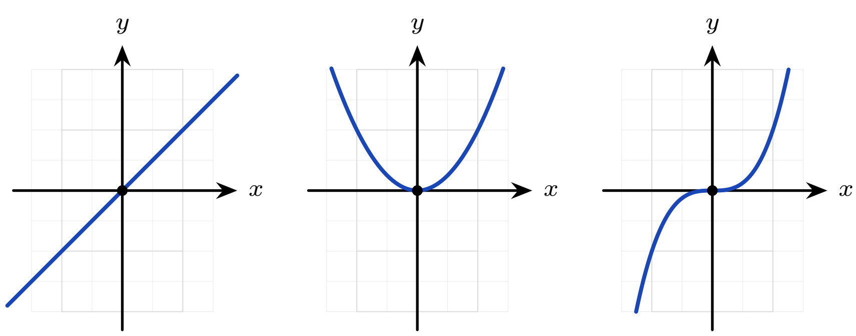
6.1.2 Roots and Multiplicity

The table gives the shape in outline. What fixes it exactly is what the curve does at each root, and that is determined by whether a factor is repeated.

The **multiplicity** of a root is the number of times its factor appears in the factorised polynomial. In $(x-1)(x-2)^2$ the root $x=1$ has multiplicity one, and the root $x=2$ has multiplicity two. A root of multiplicity one is called a **simple root**, and one of multiplicity two a **double root**.

> KEY At a root of **multiplicity one** the curve crosses the axis with a non-zero gradient. At a root of **even** multiplicity it touches the axis and turns back. At a root of **odd** multiplicity greater than one it crosses the axis, and the axis is the tangent there, so the curve is flat as it crosses.

Whether the multiplicity is odd or even determines crossing or touching. Its size determines how flat the curve is where it meets the axis.



(a) $y = x$ crosses (b) $y = x^2$ touches (c) $y = x^3$ flattens

Figure 6.1 The three behaviours at a root, shown at $x = 0$. In (a) the multiplicity is one and the curve crosses with a non-zero gradient. In (b) it is two, an even number, so the curve touches and turns back. In (c) it is three, odd, so the curve crosses, with the axis as its tangent at the root.

KEY **Sketching a factorised polynomial.**

1. At each x-intercept, read the behaviour from the multiplicity of the root.
2. Find the y-intercept by substituting $x = 0$.
3. Find the two ends from the leading term.

The case below goes from the factors to the sketch; §6.1.3 works in the other direction, from a sketch back to the factors.

Worked through. To sketch $y = (x + 1)(x - 2)^2$, read the roots and their multiplicities from the factors. The factor $(x + 1)$ is single, so the graph crosses the axis at $x = -1$. The factor $(x - 2)^2$ is squared, so the graph touches the axis at $x = 2$ and turns back. Find the y-intercept. Substitute $x = 0$: $y = (1)(-2)^2 = 4$, the point $(0, 4)$. Fix the ends from the leading term. Multiplying out leads with x^3: an odd degree with a positive coefficient, so the curve runs from $-\infty$ at the left to $+\infty$ at the right. These three facts fix the shape.

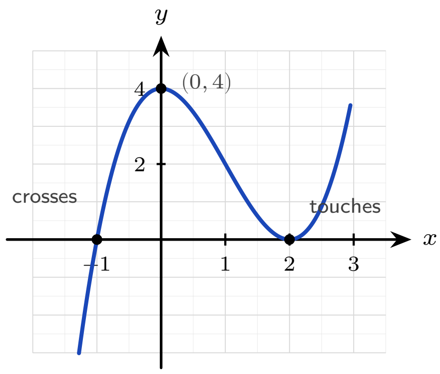


Figure 6.2 A sketch of $y = (x + 1)(x - 2)^2$. The curve crosses the axis at the simple root $x = -1$ and touches it at the double root $x = 2$. The y-intercept is $(0, 4)$, and the ends follow x^3.

When the polynomial is written out in full rather than factorised, the y-intercept is even quicker to read. Every term except the constant contains x, so substituting $x = 0$ makes all of them zero. The constant term a_0 is what remains.

KEY The graph of $y = a_n x^n + \cdots + a_1 x + a_0$ crosses the y-axis at $(0, a_0)$. *The constant term is the y-intercept.*

For $y = 2x^3 - 3x + 5$, the constant term is 5, so the graph crosses the y-axis at $(0, 5)$.

6.1.3 From the Graph Back to the Equation

The reverse problem is also examined. Given a sketch, you are asked for the polynomial that produced it. The intercepts give you almost everything you need.

Each x-intercept contributes a factor, and the behaviour there gives its multiplicity: a crossing with a non-zero gradient gives a single factor, a touch gives a squared factor, and a flat crossing gives a cubed factor. That fixes the polynomial up to one unknown number, the leading coefficient, and any known point that is not an x-intercept determines it. Check that the factors account for the whole degree of the curve; if they do not, an intercept has been missed or a multiplicity is higher than it looks.

KEY **Finding a polynomial from its graph.** Write one factor for each x-intercept, with the multiplicity the graph shows at that point. Then substitute the coordinates of any known point that is not an x-intercept to find the leading coefficient a_n.

Worked through. To find the equation of the cubic in the graph below, which touches the x-axis at $x = -1$, crosses it at $x = 2$, and passes through $(0, -1)$, use the three features the graph shows.

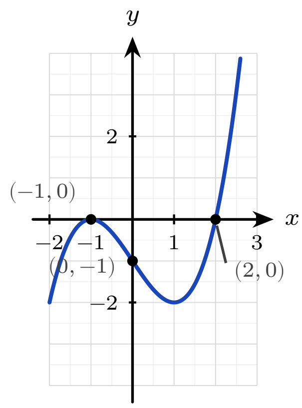


Figure 6.3 The curve touches the x-axis at $x = -1$, crosses it at $x = 2$, and passes through $(0, -1)$. These three facts determine the equation.

Write the factors from the x-intercepts. The curve touches at $x = -1$, so that root is repeated and contributes $(x + 1)^2$. It crosses at $x = 2$, so that root is single and contributes $(x - 2)$. Those factors already account for degree three, so

$$y = a(x + 1)^2(x - 2)$$

for some constant a.

Find a from the remaining point. Substitute $x = 0$ and $y = -1$:

$$-1 = a(1)^2(-2) = -2a,$$

so $a = \frac{1}{2}$ and

$$y = \tfrac{1}{2}(x + 1)^2(x - 2).$$

Check the answer against the graph. The leading term is $\frac{1}{2}x^3$, which is positive, so the curve rises to the right, and the sketch agrees.

The intercepts fix the factors; one further point fixes the leading coefficient.

YOUR TURN 6.1 Polynomial Functions and their Graphs

Degree, leading coefficient and end behaviour

1. State the degree and the leading coefficient of each polynomial.

(a) $f(x) = 3x^4 - 2x + 7$
(b) $f(x) = 5 - x^3$
(c) $f(x) = (x-1)(x+2)(x-3)$
(d) $f(x) = -2x^5 + x^2$
(e) $f(x) = (2x-1)(x+3)^2$
(f) $f(x) = (3-x)^2(2x+1)^3$

2. Describe the end behaviour of each graph as $x \to +\infty$ and as $x \to -\infty$.

(a) $y = x^3$
(b) $y = -x^4$
(c) $y = 2x^5 - x$
(d) $y = 3 - x^2$

3. Answer each question about degree and shape.

(a) State the maximum number of real roots of a degree-5 polynomial.
(b) State the maximum number of turning points of a quartic.
(c) List every possible number of turning points a cubic can have.
(d) Explain why a cubic must have at least one real root.

Roots and multiplicity

4. State the roots of each polynomial, and say at each one whether the graph crosses or touches the x-axis.

(a) $y = (x-2)(x+3)(x-5)$
(b) $y = (x-1)^2(x+4)$
(c) $y = x(x-3)^2$
(d) $y = (x+1)^2(x-2)^2$
(e) $y = (x-2)^3(x+1)$
(f) $y = x^3(x-4)^2$

5. Consider $y = (x+1)(x-3)^2$.

(a) State the roots and their multiplicities.
(b) Find the y-intercept.
(c) State the end behaviour.
(d) Sketch the curve.

From the graph to the equation

6. Find the equation of each polynomial in factorised form.

(a) A cubic crosses the x-axis at $x = -2$, $x = 1$ and $x = 3$, and passes through $(2, -8)$.

(b) A quartic touches the x-axis at $x = 1$, crosses it at $x = -1$ and $x = 3$, and passes through $(0, 6)$.

Concept check

7. A student says: "*the graph of $y = (x+1)^3(x-4)$ touches the x-axis at $x = -1$, because the factor $(x+1)$ is repeated.*" Explain why this is false.

6.2 Polynomial Division

You can already add, subtract and multiply polynomials. Addition and subtraction collect like terms, and multiplication expands every product before collecting. Division is the operation that still needs a method. Factorising depends on it: once one factor of a polynomial is known, the other is found by dividing, and §6.3 and §6.4 are built on that step.

6.2.1 The Division Algorithm

Before dividing anything it helps to know what the answer should look like, and whole-number division already shows that. When you divide 47 by 5 you write

$$47 = 5 \times 9 + 2,$$

which records a divisor, a quotient, and a remainder smaller than the divisor. Polynomial division works the same way, with "smaller" measured by degree instead of size.

KEY **The division algorithm.** For polynomials $f(x)$ and $d(x)$, with $d(x)$ not the zero polynomial, there are unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)\,q(x) + r(x),$$

where the degree of r is less than the degree of d. *Dividend equals divisor times quotient, plus remainder.*

Fix the four names now, because the methods that follow use them. The polynomial being divided, f, is the **dividend**. The polynomial you divide by, d, is the **divisor**. The result q is the **quotient**, and the part that remains, r, is the **remainder**. When $r = 0$ the division is exact, and d is a factor of f.

6.2.2 Long Division

Long division works for a divisor of any degree. It is three steps repeated until the remainder has lower degree than the divisor.

> KEY **Divide, multiply, subtract.**
>
> 1. **Divide** the leading term of the dividend by the leading term of the divisor. The result is the next term of the quotient.
> 2. **Multiply** the whole divisor by that term.
> 3. **Subtract** the product from the dividend, and repeat on what is left.

Set the work out in columns, one for each power of x: the divisor to the left of the bracket, the dividend inside it, the quotient written on the top line, and the remainder on the bottom line. Write the result as $f(x) = d(x)\,q(x) + r(x)$ when you finish, because multiplying that right-hand side out checks the whole division.

Worked through. Dividing $2x^3 - 3x^2 + x - 5$ by $(x - 2)$ sets out like this:

$$
\begin{array}{r|rrrrl}
 & & 2x^2 & +x & +3 & \text{quotient} \\ \cline{2-5}
x-2\,) & 2x^3 & -3x^2 & +x & -5 & \text{dividend} \\
 & 2x^3 & -4x^2 & & & \\ \cline{2-3}
 & & x^2 & +x & & \\
 & & x^2 & -2x & & \\ \cline{3-4}
 & & & 3x & -5 & \\
 & & & 3x & -6 & \\ \cline{4-5}
 & & & & 1 & \text{remainder}
\end{array}
$$

Each stage repeats the same three steps: divide, multiply, subtract.

Stage 1. Divide the leading term of the dividend by the leading term of the divisor: $2x^3 \div x = 2x^2$. Write $2x^2$ in the quotient. Multiply the divisor by it: $2x^2(x - 2) = 2x^3 - 4x^2$. Subtract that from the dividend, which leaves $x^2 + x$.

Stage 2. Repeat on $x^2 + x$. Divide: $x^2 \div x = x$. Multiply: $x(x - 2) = x^2 - 2x$. Subtract, which leaves $3x - 5$.

Stage 3. Repeat on $3x - 5$. Divide: $3x \div x = 3$. Multiply: $3(x - 2) = 3x - 6$. Subtract, which leaves 1.

The degree of 1 is lower than the degree of $x - 2$, so the division stops.

The quotient is $2x^2 + x + 3$ and the remainder is 1:

$$2x^3 - 3x^2 + x - 5 = (x-2)(2x^2 + x + 3) + 1.$$

CAUTION Write the dividend in descending powers and insert a zero term for every missing power before dividing. Divide $x^3 - 13x + 12$ as $x^3 + 0x^2 - 13x + 12$. Without that zero term the columns shift, and the subtractions that follow go wrong.

EXAMPLE 6.1

Divide $x^3 - 13x + 12$ by $(x-1)$.
There is no x^2 term, so write it as $x^3 + 0x^2 - 13x + 12$ and keep a column for it.

$$
\begin{array}{r|rrrr}
 & & x^2 & +x & -12 \\ \cline{2-5}
x-1 & x^3 & +0x^2 & -13x & +12 \\
 & x^3 & -x^2 & & \\ \cline{2-3}
 & & x^2 & -13x & \\
 & & x^2 & -x & \\ \cline{3-4}
 & & & -12x & +12 \\
 & & & -12x & +12 \\ \cline{4-5}
 & & & & 0
\end{array}
$$

The remainder is 0, so the division is exact and $(x-1)$ is a factor. The quotient $x^2 + x - 12$ factorises further:

$$x^3 - 13x + 12 = (x-1)(x^2 + x - 12) = (x-1)(x+4)(x-3).$$

A remainder of zero shows that the divisor is a factor. The next two sections are built on that fact.

A divisor of higher degree changes none of the three steps. What it changes is where the division stops: the remainder must have degree lower than the divisor, so dividing by a quadratic can leave a remainder with an x in it. Set the columns out in the same way, and keep a column for every power of x from the highest down to the constant, whether or not the dividend has a term there.

EXAMPLE 6.2

Divide $x^4 + 2x^2 - 3x + 1$ by $(x^2 + x - 1)$.
There is no x^3 term, so write the dividend as $x^4 + 0x^3 + 2x^2 - 3x + 1$ and give that power a

column of its own.

$$
\begin{array}{r|rrrrr}
 & & & x^2 & -x & +4 \\
\cline{2-6}
x^2+x-1 & x^4 & +0x^3 & +2x^2 & -3x & +1 \\
 & x^4 & +x^3 & -x^2 & & \\
\cline{2-4}
 & & -x^3 & +3x^2 & -3x & \\
 & & -x^3 & -x^2 & +x & \\
\cline{3-5}
 & & & 4x^2 & -4x & +1 \\
 & & & 4x^2 & +4x & -4 \\
\cline{4-6}
 & & & & -8x & +5
\end{array}
$$

Stage 1. Divide the leading terms: $x^4 \div x^2 = x^2$. Multiply the divisor by it: $x^2(x^2+x-1) = x^4+x^3-x^2$. Subtract, which leaves $-x^3+3x^2-3x$.
Stage 2. Repeat. Divide: $-x^3 \div x^2 = -x$. Multiply: $-x(x^2+x-1) = -x^3-x^2+x$. Subtract, which leaves $4x^2-4x+1$.
Stage 3. Repeat. Divide: $4x^2 \div x^2 = 4$. Multiply: $4(x^2+x-1) = 4x^2+4x-4$. Subtract, which leaves $-8x+5$.
The remainder $-8x+5$ has degree one, the divisor has degree two, so the division stops:

$$x^4+2x^2-3x+1 = (x^2+x-1)(x^2-x+4) - 8x + 5.$$

Multiplying the right side out returns the dividend, which checks every stage at once. A remainder with an x in it is the normal outcome against a quadratic divisor, not a sign that a stage has gone wrong.

YOUR TURN 6.2 Polynomial Division

Dividing by a linear factor

8. Divide, and state the quotient and the remainder.

(a) x^2+5x+6 by $(x+2)$
(b) x^3-3x^2+2x by $(x-1)$
(c) x^3+2x^2-x+1 by $(x-1)$
(d) x^3-8 by $(x-2)$

9. Divide, and state the quotient and the remainder. Insert a zero term for each missing power first.

(a) $2x^3-x^2-3$ by $(x+1)$
(b) x^3+3x^2-4 by $(x+2)$
(c) $2x^4-3x^3+x-6$ by $(x-2)$

Dividing by a quadratic

10. Divide by the quadratic divisor, and state the quotient and the remainder.

(a) $x^4 - 1$ by $(x^2 - 1)$
(b) $x^4 - 1$ by $(x^2 + 1)$
(c) $x^3 + 2x^2 - 3x + 1$ by $(x^2 + x - 1)$
(d) $2x^4 - x^3 + 3x - 2$ by $(x^2 - 2x + 2)$

The division algorithm

11. Use $f(x) = d(x)\,q(x) + r(x)$ in each part.

(a) When $f(x)$ is divided by $x^2 - 1$, the quotient is $2x + 3$ and the remainder is $x - 4$. Find $f(x)$.
(b) Divide $x^4 + x^3 - 2x + 5$ by $x^2 + 2$, and write the result in the form $f(x) = d(x)\,q(x) + r(x)$.
(c) When $x^3 + ax^2 + bx + 3$ is divided by $x^2 + 1$, the remainder is $2x + 1$. Find a and b.

Concept check

12. A student divides $x^3 + 2x^2 + x - 1$ by $x^2 + 1$ and writes: "*quotient* x, *remainder* $2x^2 - 1$." Explain why this cannot be the final answer, and find the correct quotient and remainder.

6.3 The Remainder and Factor Theorems

6.3.1 The Remainder Theorem

Here are two operations that appear unrelated. *Evaluating* f at 2 gives the height of the graph above one point. *Dividing* f by $(x - 2)$ is the long calculation of the previous section. The remainder theorem states that the two produce the same number, and it is quickly proved.

Take the division algorithm with a linear divisor $(x - a)$:

$$f(x) = (x - a)\,q(x) + r,$$

where the remainder r must be a *constant*, because its degree is less than the degree of $x - a$, and less than one means zero.

Now substitute $x = a$ into that identity. The factor $(x - a)$ becomes zero, so the whole first term is zero, leaving $f(a) = r$. The height of the graph at a *is* the remainder, because everything else in the polynomial is divisible by $(x - a)$ and so contributes nothing at that point.

KEY **Remainder theorem.** When a polynomial $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$. *One substitution replaces the division.*

The theorem holds for any linear divisor, not only for one written as $(x-a)$. Divide by $(bx-c)$ and the remainder is f evaluated at the **zero of the divisor**, meaning the value of x that makes the divisor equal to zero. For $(2x-1)$ that value is $\frac{1}{2}$, never 1 or 2.

Read the theorem in either direction. Given the coefficients it returns the remainder; given two remainders it returns two linear equations in the unknown coefficients.

Worked through. When $f(x) = x^3 - 2x + 5$ is divided by $(x-2)$, the remainder theorem gives the remainder as

$$f(2) = 2^3 - 2(2) + 5 = 8 - 4 + 5 = 9.$$

No division needed; the answer is a single evaluation.

EXAMPLE 6.3

Find the remainder when $f(x) = 8x^3 + 4x^2 - 6x + 5$ is divided by $(2x-1)$.

Evaluate at the zero of the divisor. Here $2x - 1 = 0$ when $x = \frac{1}{2}$, so the remainder is

$$f\left(\tfrac{1}{2}\right) = 8 \cdot \tfrac{1}{8} + 4 \cdot \tfrac{1}{4} - 3 + 5 = 1 + 1 - 3 + 5 = 4.$$

EXAMPLE 6.4

The polynomial $f(x) = x^3 + ax + b$ leaves remainder 7 when divided by $(x-2)$ and remainder -5 when divided by $(x+1)$. Find a and b.

Each remainder condition is one evaluation, so two conditions give two equations:

$$f(2) = 8 + 2a + b = 7 \implies 2a + b = -1,$$

$$f(-1) = -1 - a + b = -5 \implies -a + b = -4.$$

Subtracting, $3a = 3$, so $a = 1$ and $b = -3$. The remainder theorem has turned two divisions into a pair of linear equations; questions of this kind, working from the remainders back to the coefficients, are common.

The theorem is stated for a linear divisor, but the idea behind it reaches further. Divide by a quadratic and the remainder has degree at most one, so it is $rx + s$: two unknowns instead of one. The division algorithm still records the whole calculation as an identity, and each zero of the divisor makes the divisor term vanish, leaving one equation in r and s. Two distinct zeros give two equations, which is exactly what two unknowns need. The method needs the two zeros to be distinct. Against a repeated factor such as $(x-1)^2$ only one substitution is available, so take the second equation from comparing coefficients.

EXAMPLE 6.5

Find the remainder when $f(x) = x^3 + 2x^2 - 5x + 3$ is divided by $(x-1)(x-2)$.

The divisor has degree two, so the remainder has degree at most one. Write it as $rx + s$ and record the division as an identity:

$$f(x) = (x-1)(x-2)\,q(x) + rx + s.$$

This holds for every value of x, so choose the two values that make the product term disappear. Substituting $x = 1$,

$$f(1) = 1 + 2 - 5 + 3 = 1 \implies r + s = 1,$$

and substituting $x = 2$,

$$f(2) = 8 + 8 - 10 + 3 = 9 \implies 2r + s = 9.$$

Subtracting the first equation from the second gives $r = 8$, and then $s = -7$. The remainder is

$$8x - 7.$$

Long division confirms it: the quotient is $x + 5$, and $(x-1)(x-2)(x+5) + 8x - 7$ returns $f(x)$. Both substitutions were available because the zeros 1 and 2 are distinct.

6.3.2 The Factor Theorem

More follows when the remainder is zero. If dividing by $(x - a)$ leaves no remainder, then $(x - a)$ divides $f(x)$ exactly: it is a factor. That is what turns a test for a root into a method for factorising.

KEY **Factor theorem.** $(x - a)$ is a factor of $f(x)$ if and only if $f(a) = 0$.

To factorise an unfamiliar cubic, search for a root by testing small values. The factors of the constant term are the natural candidates, and the reason is short. If a is an integer root of $a_n x^n + \cdots + a_1 x + a_0$ with integer coefficients, then substituting and moving a_0 across gives

$$a\left(a_n a^{n-1} + a_{n-1} a^{n-2} + \cdots + a_1\right) = -a_0,$$

so a non-zero integer root a divides a_0. When the leading coefficient is not 1, the candidates are the fractions $\frac{p}{q}$ with p a factor of a_0 and q a factor of a_n. Once one root a is found, $(x - a)$ is a factor, and dividing it out leaves a polynomial one degree lower; for a cubic, that is a quadratic you can finish by hand. For a root $\frac{p}{q}$ the factor is $(qx - p)$.

That division need not be long division. To **compare coefficients**, write the quotient with unknown coefficients, one degree lower than the dividend, multiply out, and equate the coefficients of matching powers. Each unknown comes from one comparison, and any coefficient left over is a check.

Worked through. To show that $(x-1)$ is a factor of $f(x) = x^3 - 7x^2 + 14x - 8$, and to factorise f completely, test $x = 1$:

$$f(1) = 1 - 7 + 14 - 8 = 0,$$

so by the factor theorem $(x-1)$ is a factor.

Now divide it out. Instead of long division, *compare coefficients*. The quotient must be a quadratic, so write

$$x^3 - 7x^2 + 14x - 8 = (x-1)(x^2 + bx + c),$$

then compare the coefficients of matching powers on the two sides. Compare the constant terms: $-c = -8$, so $c = 8$. Compare the coefficients of x^2: $b - 1 = -7$, so $b = -6$. The coefficient of x has not been used, so use it as a check: on the right it is $c - b = 8 + 6 = 14$, which matches the left. ✓ Hence

$$f(x) = (x-1)(x^2 - 6x + 8) = (x-1)(x-2)(x-4).$$

Had the factor not been given, the constant term -8 would give the candidates $\pm1, \pm2, \pm4, \pm8$, and $x = 1$ is the first to test.

TIP Comparing coefficients is often quicker than long division for a linear divisor. Long division is equally acceptable if you prefer it.

YOUR TURN 6.3 The Remainder and Factor Theorems

The remainder theorem

13. Find the remainder in each case.

(a) $x^3 + 2x - 1$ divided by $(x - 1)$

(b) $x^3 - 4x^2 + x + 6$ divided by $(x - 2)$

(c) $2x^3 - 3x + 1$ divided by $(x + 1)$

(d) $x^3 + x^2 + x + 1$ divided by $(x + 2)$

14. Find the remainder in each case. The divisor is not of the form $(x - a)$.

(a) $4x^3 - 4x^2 + 5x - 1$ divided by $(2x - 1)$

(b) $2x^3 + x^2 - 2$ divided by $(2x + 1)$

15. Use the remainder and factor theorems to answer each part.

(a) The remainder when $f(x)$ is divided by $(x - 3)$ is 0. State what this tells you about f.

(b) The remainder when $f(x)$ is divided by $(x - 2)$ is 7. State $f(2)$.

(c) $f(-1) = 0$. Write down a factor of $f(x)$.

(d) $x^4 - 1$ is divided by $(x - 1)$. Find the remainder without dividing.

16. The polynomial $f(x) = x^3 + ax^2 + bx + 6$ has $(x + 1)$ as a factor, and leaves remainder 12 when divided by $(x - 2)$.

(a) Write down two equations in a and b.

(b) Solve them to find a and b.

The remainder on division by a quadratic

17. The remainder on division by a quadratic $(x - a)(x - b)$ has the form $rx + s$. Find it in each case.

(a) $x^3 - 2x^2 + 4x - 1$ divided by $(x - 1)(x - 3)$

(b) $x^4 + x - 3$ divided by $(x + 1)(x - 2)$

The factor theorem

18. Decide whether the given linear expression is a factor. Show your working.

(a) Is $(x-2)$ a factor of x^3-5x^2+8x-4?

(b) Is $(x+3)$ a factor of x^3+x^2-5x+3?

(c) Is $(x-1)$ a factor of x^3+x^2+x+1?

(d) Is $(x+2)$ a factor of x^3+8?

19. Factorise each polynomial completely.

(a) x^3-x

(b) x^3-4x

(c) $x^3-3x^2-4x+12$

(d) $2x^3+3x^2-11x-6$

20. In each part, $(x-3)$ is a factor. Find the quadratic factor by comparing coefficients, then factorise completely.

(a) x^3-2x^2-5x+6

(b) $2x^3-5x^2-4x+3$

21. Use the factor theorem in each part.

(a) One factor of x^2-5x+6 is $(x-2)$. State the other.

(b) Show that x^3-3x^2+4 has a repeated factor.

Concept check

22. Each student below has made an error in using the remainder theorem. Identify the error and find the correct remainder.

(a) To find the remainder when $f(x)=x^3-2x+5$ is divided by $(x+3)$, a student calculates $f(3)=26$.

(b) To find the remainder when $g(x)=4x^3-2x+3$ is divided by $(2x-1)$, a student calculates $g(1)=5$.

6.4 Solving Polynomial Equations

You already know how to solve linear and quadratic equations. A linear equation is rearranged directly. A quadratic is solved by factorising, by completing the square, or by the quadratic formula.

For degrees three and four, general formulas exist but are far too long to use by hand, and from degree five upward no general formula exists. The factor theorem takes their place when there is a rational root to find.

KEY To solve a polynomial equation of degree 3 or more:

1. Test the factors of the constant term, or the fractions $\frac{p}{q}$ of §6.3.2 if the leading coefficient is not 1, until one value a gives $f(a) = 0$.
2. Divide $f(x)$ by $(x - a)$, either by long division or by comparing coefficients.
3. Repeat on the quotient until a quadratic remains, then solve that in the usual way.
4. If no candidate is a root, find the roots with a GDC.

Every real root is an x-intercept of the graph, so a sketch or a GDC graph shows how many real roots to look for.

Worked through. To solve $x^3 - 3x^2 - x + 3 = 0$, test the factors of 3. At $x = 1$: $1 - 3 - 1 + 3 = 0$, so $(x - 1)$ is a factor. Dividing,

$$x^3 - 3x^2 - x + 3 = (x - 1)(x^2 - 2x - 3) = (x - 1)(x - 3)(x + 1).$$

The roots are $x = 1,\ 3,\ -1$.

A cubic with real coefficients always has at least one real root, because one end goes to $+\infty$ and the other to $-\infty$, so the curve must cross the axis. If none of the candidates is a root, the cubic has no rational roots, so every real root is irrational; the equation is then solved with a GDC.

YOUR TURN 6.4 Solving Polynomial Equations

Equations that factorise at sight

23. Solve each equation. Each is already factorised or factorises at sight.

(a) $(x - 1)(x + 2)(x - 4) = 0$ (b) $(x - 3)^2(x + 1) = 0$

(c) $x^3 = 4x$ (d) $x^3 + x^2 - 2x = 0$

Using the factor theorem

24. Solve each equation, using the factor theorem to find the first root.

(a) $x^3 + 2x^2 - 5x - 6 = 0$ (b) $x^3 - 3x^2 - 10x + 24 = 0$

(c) $2x^3 - 3x^2 - 3x + 2 = 0$

25. Solve $x^3 - 1 = 0$, giving only the real root, and explain why there is exactly one.

Rational roots

26. The leading coefficient is not 1, so test the candidates $\frac{p}{q}$. Find one root, then solve the equation.

(a) $2x^3 - x^2 + 4x - 2 = 0$
(b) $12x^3 - 16x^2 - 5x + 3 = 0$

Quartic equations

27. Solve each equation.

(a) $x^4 - 5x^2 + 4 = 0$ *(Hint: substitute $u = x^2$)*
(b) $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$ *(Hint: use the factor theorem twice)*

When no candidate is a root

28. GDC For each equation, show that none of the candidate integer roots is a root. Then use a GDC to find every real root, correct to three significant figures.

(a) $x^3 - 4x + 1 = 0$
(b) $x^3 + 2x^2 - 2 = 0$

Concept check

29. To solve $x^3 + 6x^2 + 11x + 6 = 0$, a student tests $x = 1, 2, 3$ and 6, finds that none is a root, and concludes: "*there are no integer roots, so a GDC is needed.*" Identify the error, and solve the equation.

6.5 Sum and Product of Roots

The coefficients of a polynomial determine the sum and the product of its roots. You can therefore find those two values *without solving the equation at all*. This matters whenever the roots are irrational, or when the equation is too hard to solve, because the sum and the product remain easy to obtain.

6.5.1 The Quadratic Case

Ch. 4 showed that if $ax^2 + bx + c = 0$ has roots α and β, then comparing $ax^2 + bx + c$ with $a(x - \alpha)(x - \beta)$ gives

$$\alpha + \beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{c}{a}.$$

The minus sign on the sum comes from the brackets, one $-\alpha$ and one $-\beta$.

6.5.2 The General Case

The same comparison works for every degree, using the counting argument of Ch. 2. Expanding

$$a_n(x - r_1)(x - r_2)\cdots(x - r_n)$$

means choosing, from each bracket, either the x or the root, exactly as in the binomial theorem. To build the x^{n-1} term you take x from all brackets but one, so each root appears exactly once, each carrying a single minus sign: the coefficient equals $-(r_1 + r_2 + \cdots + r_n)$, the *sum*. To build the constant term you take the root from *every* bracket: all n roots multiplied together, carrying n minus signs, which is where the $(-1)^n$ comes from. So the leading coefficient, the next coefficient and the constant term give the sum and the product of the roots.

KEY · IB For the equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$,

$$\text{sum of roots} = -\frac{a_{n-1}}{a_n}, \qquad \text{product of roots} = (-1)^n\,\frac{a_0}{a_n}.$$

Divide by the leading coefficient. The sum is the next coefficient with its sign changed; the product is the constant term, with sign $(-1)^n$.

These are the first and last of a family of relations (Vieta's formulas), but the sum and product are the two the syllabus requires, used for checking answers, for building equations with required roots, and for finding an unknown coefficient.

One more relation fills the gap between them. Building the x^{n-2} term means taking the root from exactly two brackets and x from the rest, so that coefficient is the sum of the roots multiplied together in pairs. For a cubic $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ with roots α, β, γ, the three relations together are

$$\alpha + \beta + \gamma = -\frac{a_2}{a_3}, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{a_1}{a_3}, \qquad \alpha\beta\gamma = -\frac{a_0}{a_3}.$$

The syllabus requires only the first and the last. The middle relation is a short extension, and it is needed for expressions such as $\alpha^2 + \beta^2 + \gamma^2$, which involve the roots in pairs.

EXAMPLE 6.6

The equation $2x^2 - 5x + 3 = 0$ has roots α and β. Find $\alpha + \beta$ and $\alpha\beta$ without solving, then verify.

Here $a = 2$, $b = -5$, $c = 3$, so

$$\alpha + \beta = -\frac{-5}{2} = \frac{5}{2}, \qquad \alpha\beta = \frac{3}{2}.$$

Solving confirms it: the roots are 1 and $\frac{3}{2}$, whose sum is $\frac{5}{2}$ and product is $\frac{3}{2}$. The sum and product were available from the coefficients before the roots were found.

Worked through. The cubic $x^3 - 2x^2 - 9x + 18 = 0$ has roots α, β, γ. With $a_3 = 1$, $a_2 = -2$, $a_0 = 18$ and degree $n = 3$,

$$\alpha + \beta + \gamma = -\frac{-2}{1} = 2, \qquad \alpha\beta\gamma = (-1)^3 \frac{18}{1} = -18.$$

(The roots happen to be $-3, 2, 3$: sum 2, product -18, as promised.)

6.5.3 Symmetric Combinations of the Roots

The sum and the product are more useful than they appear, because many other expressions in the roots can be rebuilt from them. That is what makes the method valuable when the roots are irrational: the worked case and the examples below answer precise questions about the roots without ever finding one.

A **symmetric combination** of the roots is an expression whose value does not change when the roots are exchanged. Both $\alpha^2 + \beta^2$ and $\frac{1}{\alpha} + \frac{1}{\beta}$ are symmetric; $\alpha - \beta$ is not. Every symmetric combination of two roots can be written in terms of $\alpha + \beta$ and $\alpha\beta$. Two rearrangements cover most cases, and a sum of reciprocals reduces by putting it over a common denominator, $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$.

KEY

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta, \qquad (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta.$$

Neither is a new formula. Each is the expansion of $(\alpha \pm \beta)^2$ rearranged.

Worked through. The equation $x^2 - 3x + 1 = 0$ has roots α and β. To find $\alpha^2 + \beta^2$ and $\frac{1}{\alpha} + \frac{1}{\beta}$ without solving, work from the sum and the product.

The roots are irrational, but their sum and product are not: $\alpha + \beta = 3$ and $\alpha\beta = 1$. For the sum of squares, use the first rearrangement above:

$$(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 \implies \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 9 - 2 = 7.$$

For the sum of reciprocals, combine over a common denominator:

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{3}{1} = 3.$$

Neither answer required a root. Both were rebuilt from the sum and the product alone.

Reading the relations backwards builds an equation from its roots. A quadratic whose roots have sum S and product P is

$$x^2 - Sx + P = 0,$$

so an equation whose roots are a transformation of α and β is found by transforming the sum and the product. The roots themselves are never needed.

EXAMPLE 6.7

With α, β as above, find a quadratic equation with integer coefficients whose roots are 2α and 2β.

A quadratic is determined by the sum and product of its roots: $x^2 - (\text{sum})x + (\text{product}) = 0$. The new sum and product follow from the old ones:

$$2\alpha + 2\beta = 2(\alpha + \beta) = 6, \qquad (2\alpha)(2\beta) = 4\alpha\beta = 4.$$

So the required equation is

$$x^2 - 6x + 4 = 0.$$

No root was ever computed: transforming an equation's roots means transforming their sum and product, nothing more.

EXAMPLE 6.8

The roots of $x^2 + kx + 12 = 0$ differ by 1. Find the possible values of k.

Let the roots be α and β with $\alpha - \beta = 1$. The coefficients give $\alpha + \beta = -k$ and $\alpha\beta = 12$. Connect the difference to these totals by squaring it:

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta \implies 1 = k^2 - 48,$$

so $k^2 = 49$ and $k = \pm 7$. Check with $k = 7$: the equation $x^2 + 7x + 12 = 0$ has roots -3 and -4, which indeed differ by 1. A condition on the roots has become an equation in the coefficient.

YOUR TURN 6.5 Sum and Product of Roots

Sum and product from the coefficients

30. State the sum and the product of the roots of each equation.

(a) $x^2 - 7x + 10 = 0$

(b) $2x^2 + 3x - 5 = 0$

(c) $x^2 + 4x + 4 = 0$

(d) $3x^2 - 2x - 8 = 0$

31. For each cubic or quartic, state the sum and the product of the roots.

(a) $x^3 - 6x^2 + 11x - 6 = 0$

(b) $2x^3 - 4x^2 + x - 3 = 0$

(c) $x^4 - 3x^3 + 2x - 5 = 0$

Symmetric expressions in the roots

32. The roots of $2x^2 - 7x + 6 = 0$ are α and β. Without solving the equation, find each value.

(a) $\alpha + \beta$ and $\alpha\beta$

(b) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$

(c) $\alpha^2 + \beta^2$

(d) $(\alpha - \beta)^2$

33. The roots of $x^3 - 4x^2 + x + 6 = 0$ are α, β and γ.

(a) Write down $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$.

(b) Write down $\alpha\beta + \beta\gamma + \gamma\alpha$.

(c) Hence find $\alpha^2 + \beta^2 + \gamma^2$.

Building an equation from its roots

34. Build the required equation.

(a) Write a quadratic with leading coefficient 1 whose roots sum to 5 and multiply to 6.

(b) Write a cubic with leading coefficient 1 whose roots are -1, 2 and 4.

35. The roots of $x^2 + 5x + 3 = 0$ are α and β.

(a) Write down $\alpha + \beta$ and $\alpha\beta$.

(b) Find the sum and product of $\alpha + 2$ and $\beta + 2$.

(c) Hence find a quadratic equation with integer coefficients whose roots are $\alpha + 2$ and $\beta + 2$.

Conditions on the roots

36. In each part, a condition on the roots fixes an unknown coefficient.

(a) The roots of $x^2 + bx + 12 = 0$ are 3 and one other. Find the other root and b.

(b) The roots of $x^2 - 7x + k = 0$ differ by 3. Find k.

(c) One root of $x^2 + kx + 18 = 0$ is twice the other. Find the possible values of k.

Concept check

37. A student writes: "*the product of the roots of* $2x^3 - 5x^2 + x + 6 = 0$ *is* $\dfrac{6}{2} = 3$, *as for a quadratic.*" Identify the error and state the correct product.

6.6 Partial Fractions

Adding two simple fractions produces one more complicated fraction. **Partial fractions** reverse that process: they split a single fraction back into a sum of simpler ones, one for each factor of the denominator. The split form is easier to use, and it makes a rational function easier to integrate. See Ch. 15.

The syllabus limits the problem to a denominator of **at most two distinct linear factors**, with the numerator of lower degree than the denominator. That is the case treated below. The larger theory is sketched at the end of the section for reference.

Each linear factor contributes one term with a constant numerator.

KEY For a proper fraction over two distinct linear factors,

$$\frac{px+q}{(x-\alpha)(x-\beta)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta}.$$

The same form holds when a factor is written as $(ax+b)$, for example $\dfrac{A}{2x-1}$; only the value of x that makes that factor zero changes.

To find A and B, multiply through by the denominator, then substitute the values of x that make one factor zero. Each substitution isolates one unknown.

Worked through. To express $\dfrac{3x+5}{(x-1)(x+2)}$ in partial fractions, write

$$\frac{3x+5}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2},$$

and multiply through by $(x-1)(x+2)$:

$$3x+5 = A(x+2) + B(x-1).$$

Setting $x=1$ isolates A: $8 = 3A$, so $A = \frac{8}{3}$. Setting $x=-2$ isolates B: $-1 = -3B$, so $B = \frac{1}{3}$. Therefore

$$\frac{3x+5}{(x-1)(x+2)} = \frac{8}{3(x-1)} + \frac{1}{3(x+2)}.$$

Choosing the two values that each make one factor zero is what makes the method quick. It turns a pair of simultaneous equations into two one-line calculations.

EXAMPLE 6.9

Express $\dfrac{x+5}{x^2-x-2}$ in partial fractions.

This denominator is not factorised, so factorise it first:

$$x^2 - x - 2 = (x-2)(x+1).$$

Now proceed as before:

$$\frac{x+5}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \implies x+5 = A(x+1) + B(x-2).$$

Setting $x = 2$: $7 = 3A$, so $A = \frac{7}{3}$. Setting $x = -1$: $4 = -3B$, so $B = -\frac{4}{3}$. Hence

$$\frac{x+5}{x^2-x-2} = \frac{7}{3(x-2)} - \frac{4}{3(x+1)}.$$

TIP A question may give the denominator in expanded form. The factorising step is part of the answer, so write it out.

EXT **Beyond the syllabus.** Three things extend the method. None of them is on the AA HL syllabus, although the first is short enough to try in Your Turn Q42. A fraction whose numerator has degree equal to or higher than its denominator is **improper**, and must be divided first: $\frac{x^2}{x^2-1} = 1 + \frac{1}{x^2-1}$, and only the proper part is split. A **repeated** factor $(x-\alpha)^2$ needs one term for each power, $\frac{A}{x-\alpha} + \frac{B}{(x-\alpha)^2}$. An **irreducible quadratic**, one with negative discriminant, takes a linear numerator $\frac{Bx+C}{x^2+bx+c}$. One rule covers the last two: give each factor a numerator of degree one less than itself, and an extra term for every power of a repeated factor.

Factor in the denominator	Term(s) it contributes
linear $(x-\alpha)$	$\frac{A}{x-\alpha}$
repeated linear $(x-\alpha)^2$	$\frac{A}{x-\alpha} + \frac{B}{(x-\alpha)^2}$
irreducible quadratic (x^2+bx+c)	$\frac{Bx+C}{x^2+bx+c}$

YOUR TURN 6.6 Partial Fractions

Two linear factors

38. Express each fraction in partial fractions.

(a) $\dfrac{1}{(x-1)(x-2)}$ (b) $\dfrac{5}{(x-2)(x+3)}$

(c) $\dfrac{3}{(x+1)(x+4)}$ (d) $\dfrac{x}{(x-1)(x+1)}$

39. Express each fraction in partial fractions.

(a) $\dfrac{2x+1}{x(x+1)}$ (b) $\dfrac{x+4}{x(x-2)}$

(c) $\dfrac{2x}{(x-3)(x+1)}$ (d) $\dfrac{5x-1}{(x-1)(x+2)}$

Factorising the denominator first

40. Factorise the denominator first, then express each fraction in partial fractions.

(a) $\dfrac{4x}{x^2-4}$ (b) $\dfrac{x+7}{x^2+x-2}$

A factor of the form $ax+b$

41. Express each fraction in partial fractions. Factorise the denominator first where it is not already factorised.

(a) $\dfrac{5}{(2x+1)(x-2)}$ (b) $\dfrac{5x+1}{(3x-1)(x+1)}$

(c) $\dfrac{4x-7}{2x^2-5x+3}$

An improper fraction

42. Consider $\dfrac{x^2}{x^2-1}$.

(a) Explain why this fraction is improper.

(b) Divide to write it as $1+\dfrac{1}{x^2-1}$.

(c) Hence express the whole fraction in partial fractions.

Concept check

43. A student writes $\dfrac{2x^2+1}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$, substitutes $x=1$ and $x=-2$, and obtains $A=1$ and $B=-3$. Explain why the result is wrong, and give the correct partial fractions.

Chapter 6 · Polynomials

Reflections

TOK Abel and Galois proved something surprising. The roots of a general quintic *exist*, but they cannot be written using only addition, subtraction, multiplication, division and roots. The equation $x^5 - x - 1 = 0$ has a real solution. That number is completely determined, and a computer can calculate it to a thousand decimal places. Even so, no formula of that kind can express it exactly. What does it mean, then, to "know" a number? Is such a root any less real than one you can write down?

IM The history behind this chapter is unusual. In Italy in the 1530s, mathematicians competed in public problem-solving contests, and a loser could lose his university post, so methods were kept secret. Scipione del Ferro solved one type of cubic around 1515, Niccolò Tartaglia rediscovered the method in 1535, and Gerolamo Cardano published it in 1545, after promising Tartaglia he would not, together with his student Ferrari's solution of the quartic. Almost three hundred years later Niels Abel proved that no such formula exists for the general quintic, and Évariste Galois, killed in a duel at twenty, explained exactly which equations a formula can solve. To do so he created *group theory*, the mathematical study of symmetry, which is now used in particle physics and cryptography.

RW Polynomials are widely used in computing, because they need only multiplication and addition, which processors carry out very quickly. The curved shapes in fonts and animation are drawn with *Bézier curves*, which are polynomial curves whose shape is controlled by a few chosen points. *Error-correcting codes* are built from polynomials as well. These are methods of adding extra information to a message so that it can still be read after part of it is damaged, which is how a scratched disc still plays and how a faint signal from a spacecraft is recovered. Polynomials are also used to fit smooth models to scattered measurements in science and finance.

EXT The factor theorem raises a question it cannot answer. How many roots does a polynomial really have? Over the real numbers, $x^2 + 1$ has none. The **Fundamental Theorem of Algebra** gives the answer. Over the *complex* numbers of Chapter 17, every polynomial of degree n has exactly n roots, counted with multiplicity. The sum and product formulas then hold for every polynomial, because the roots they describe always exist.

End-of-Chapter Problems

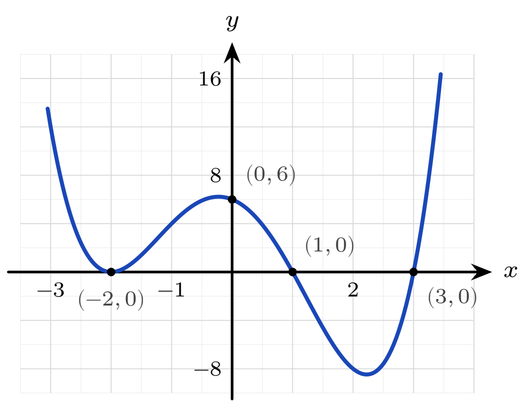
1. Sketch each graph, showing all intercepts and describing the behaviour at each root.

(a) $y = (x+1)(x-1)(x-3)$ [3]

(b) $y = x^2(x-3)$ [3]

(c) $y = -(x+1)^3(x-2)$ [3]

2. The graph of a polynomial $y = f(x)$ of degree four is shown. It touches the x-axis at $x = -2$, crosses it at $x = 1$ and $x = 3$, and meets the y-axis at $(0, 6)$.



(a) Find an expression for $f(x)$ in factorised form. [3]

(b) Hence solve the inequality $f(x) \leq 0$. [2]

3. Let $f(x) = 2x^4 - 3x^3 + 3x^2 + 8x - 1$.

(a) Find the quotient and the remainder when $f(x)$ is divided by $x^2 - 2x + 3$. [4]

(b) Hence find the values of the constants a and b for which $f(x) + ax + b$ is exactly divisible by $x^2 - 2x + 3$. [2]

4. The polynomial $x^3 - 3x^2 + kx - 8$ has $(x-2)$ as a factor. Find k. [3]

5. When a polynomial $p(x)$ is divided by $(x+1)$ the remainder is 7, and when it is divided by $(x-3)$ the remainder is -1. Find the remainder when $p(x)$ is divided by $x^2 - 2x - 3$. [3]

6. Find the unknown coefficients in each polynomial.

(a) The polynomial $x^3 + ax^2 + bx - 2$ has $(x-1)$ and $(x-2)$ as factors. Find a and b. [3]

(b) When $4x^3 + cx + d$ is divided by $(2x-1)$ the remainder is 6, and $(x+1)$ is a factor. Find c and d. [4]

7. Build the equation described in each part.

(a) A quadratic $x^2 + px + q$ has roots 3 and -7. Find p and q. [2]

(b) The cubic $x^3 + px^2 + qx + r$ has roots -1, 2 and 4. Find p, q and r. [3]

(c) The quadratic equation with roots α and β is $x^2 - 4x + 1 = 0$. Find a quadratic with leading coefficient 1 whose roots are $\alpha + 1$ and $\beta + 1$. [4]

8. The roots of $x^2 + bx + 12 = 0$ differ by 1. Use the sum and product of the roots to find the possible values of b. [3]

9. Each equation has roots α and β. Answer without solving the equation.

(a) $3x^2 - x - 2 = 0$: find $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$. [2]

(b) $x^2 - 6x + 7 = 0$: find $\alpha^2 + \beta^2$ and $(\alpha - \beta)^2$. [3]

10. The roots of $x^3 + 2x^2 + 3x - 4 = 0$ are α, β and γ.

(a) Find $\alpha^2 + \beta^2 + \gamma^2$ without solving the equation. [3]

(b) Hence show that the three roots are not all real. [1]

11. Solve $3x^3 - 4x^2 - 5x + 2 = 0$. [4]

12. Let $f(x) = 2x^3 - x^2 - 13x - 6$.

(a) Show that $(x + 2)$ is a factor of $f(x)$. [1]

(b) Factorise $f(x)$ completely. [3]

(c) Hence solve $2e^{3y} - e^{2y} - 13e^y - 6 = 0$. [2]

13. The polynomial $x^3 - kx + 2$ has $(x - 2)$ as a factor.

(a) Find k. [2]

(b) Find the other two roots, in exact form. [4]

14. The equation $2x^3 - 3x^2 + kx + 6 = 0$ has roots α, $-\alpha$ and β. Find β, the value of k and the value of α^2. [5]

15. Consider $f(x) = x^4 - x^2 - 12$.

(a) Use the substitution $u = x^2$ to find the real solutions of $f(x) = 0$. [3]

(b) Explain why $f(x) = 0$ has no other real solutions. [2]

(c) Factorise $f(x)$ as far as possible over the real numbers. [2]

16. Consider $f(x) = 12x^3 - 16x^2 - 7x + 6$.

(a) Any rational root of f has the form $\dfrac{p}{q}$ in lowest terms. State the conditions on p and q. [2]
(b) Show that $x = \frac{1}{2}$ is a root. [2]
(c) Hence factorise $f(x)$ completely. [3]
(d) Solve $f(x) = 0$. [1]

17. Consider $f(x) = x^3 - 3x + 2$.

(a) Show that $(x - 1)$ is a factor, and factorise $f(x)$ completely. [4]
(b) Sketch $y = f(x)$, showing all intercepts and the behaviour at each root. [3]
(c) Hence solve the inequality $f(x) > 0$. [2]

18. When $f(x) = x^3 + ax^2 + bx + c$ is divided by $x^2 - 1$ the remainder is $2x - 1$, and $(x - 2)$ is a factor of $f(x)$.

(a) Find the values of a, b and c. [4]
(b) Hence solve $f(x) = 0$, giving your answers in exact form. [3]

19. Express each fraction in partial fractions.

(a) $\dfrac{5x + 5}{x^2 - x - 6}$ [3]
(b) $\dfrac{7x + 1}{2x^2 + x - 1}$ [3]

20.

(a) Express $\dfrac{2}{(2r - 1)(2r + 1)}$ in partial fractions. [2]
(b) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{2}{(2r - 1)(2r + 1)} = \frac{2n}{2n + 1}$. [3]
(c) Find the least value of n for which the sum in part (b) is greater than 0.99. [2]

21. GDC Let $f(x) = x^3 - 5x^2 + 2x + 7$.

(a) Show that $f(x) = 0$ has no rational roots. [2]
(b) Use your GDC to find the three roots of $f(x) = 0$, giving your answers to three significant figures. [2]
(c) Hence solve the inequality $f(x) < 0$. [2]
(d) Verify that your answers to part (b) are consistent with the sum of the roots. [1]

22. GDC An open box is made from a rectangular sheet of card 30 cm by 20 cm by cutting a square of side x cm from each corner and folding up the sides.

(a) Show that the volume of the box is $V(x) = 4x^3 - 100x^2 + 600x$ cm^3. [2]
(b) State the set of possible values of x. [1]

(c) Find the maximum volume of the box, and the value of x that gives it. [2]
(d) Find the values of x for which the volume is 800 cm^3. [3]

Give your answers to parts (c) and (d) to three significant figures.

23. Prove that if a polynomial with integer coefficients has a rational root $\dfrac{p}{q}$ in lowest terms, then p divides the constant term and q divides the leading coefficient. [6]

24. The cubic $x^3 + px + q = 0$ has a repeated root. Show that $4p^3 + 27q^2 = 0$. [6]

25. Find all real values of k for which $x^3 - 3x + k = 0$ has three distinct real roots. *(Hint: use Q24 to find the values of k that give a repeated root)* [6]

26. The polynomial $p(x) = x^3 + ax^2 + bx + 8$ leaves a remainder of 20 when divided by $(x - 3)$, and $(x + 2)$ is a factor.

(a) Show that $3a + b = -5$ and $2a - b = 0$. [4]
(b) Hence find a and b. [2]
(c) Factorise $p(x)$ as far as possible over the real numbers. [3]
(d) State how many real roots $p(x) = 0$ has, giving a reason. [2]
(e) Find the remainder when $p(x)$ is divided by $(x - 1)(x + 2)$. *(Hint: the remainder has the form $rx + s$)* [3]

27. Consider $f(x) = 2x^3 - 3x^2 - 11x + 6$.

(a) Show that $(x - 3)$ is a factor of $f(x)$. [2]
(b) Factorise $f(x)$ completely. [4]
(c) Solve $f(x) = 0$. [2]
(d) State the y-intercept, and describe the behaviour of $f(x)$ as $x \to \pm\infty$. [3]
(e) Sketch $y = f(x)$, showing all intercepts. [2]
(f) Hence solve the inequality $f(x) \leq 0$. [3]

28. The cubic equation $x^3 + px^2 + qx + r = 0$ has roots α, β and γ.

(a) Write down $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$ in terms of p, q and r. [3]
(b) The equation $x^3 - 6x^2 + 3x + 10 = 0$ has roots α, β, γ. Find $\alpha^2 + \beta^2 + \gamma^2$ without solving the equation. [4]
(c) Find $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. [3]
(d) Given that one root is -1, find the other two. [3]
(e) Write down a cubic equation with integer coefficients whose roots are 2α, 2β and 2γ. [3]

29. GDC A section of a roller-coaster track is modelled by $h(x) = ax^3 + bx^2 + cx + d$, $0 \leq x \leq 30$, where h m is the height of the track above the ground at a horizontal distance of x m from the start of the section. The track is 20 m high at the start, 34 m high when $x = 10$, 18 m high when $x = 20$ and 26 m high at the end of the section.

(a) Write down four equations in a, b, c and d. [2]
(b) Use your GDC to find the values of a, b, c and d. [2]
(c) Find the greatest and least heights of the track in this section. [3]
(d) Find the set of values of x for which the track is more than 25 m above the ground. [3]

Give your answers to parts (c) and (d) to three significant figures.

Challenge Questions

30. Power sums of the roots. The equation $x^2 - px + q = 0$ has roots α and β. For each integer $k \geq 0$ define

$$S_k = \alpha^k + \beta^k.$$

(a) Write down the values of S_0 and S_1. [2]
(b) Show that $S_2 = p^2 - 2q$. [2]
(c) Find S_3 in terms of p and q. [3]
(d) Explain why $\alpha^k = p\,\alpha^{k-1} - q\,\alpha^{k-2}$ for every $k \geq 2$, and deduce that $S_k = p\,S_{k-1} - q\,S_{k-2}$. [4]
(e) The equation $x^2 - 3x + 1 = 0$ has roots α and β. Use the recurrence to find S_2, S_3 and S_4. [4]
(f) The roots of $x^2 - 3x + 1 = 0$ are irrational. Explain why S_k is nevertheless an integer for every k. [3]
(g) For $x^2 - x - 1 = 0$ the numbers S_k are the *Lucas numbers* $2, 1, 3, 4, 7, \ldots$. Now let the cubic $x^3 - px^2 + qx - r = 0$ have roots α, β and γ, and let $S_k = \alpha^k + \beta^k + \gamma^k$. Find a recurrence for S_k in the style of part (d), and suggest how the investigation could continue. [3]

31. Palindromic polynomials. A polynomial is *palindromic* if its list of coefficients reads the same forwards as backwards. For example $x^4 + x^3 - 4x^2 + x + 1$ has coefficients $1,\ 1,\ -4,\ 1,\ 1$.

(a) Explain why $x = 0$ is never a root of a palindromic polynomial. [2]
(b) Let $P(x)$ be palindromic of degree n. Show that $x^n P\left(\dfrac{1}{x}\right) = P(x)$, and deduce that if α is a root of P then so is $\dfrac{1}{\alpha}$. [4]
(c) Divide $x^4 + x^3 - 4x^2 + x + 1 = 0$ through by x^2 and substitute $u = x + \dfrac{1}{x}$. Show that the equation becomes $u^2 + u - 6 = 0$. [4]
(d) Hence find all four roots of $x^4 + x^3 - 4x^2 + x + 1 = 0$ in exact form. [5]

(e) Show that every palindromic polynomial of odd degree has $x = -1$ as a root. [3]

(f) Show that $x^3 + \dfrac{1}{x^3} = u^3 - 3u$, where $u = x + \dfrac{1}{x}$. Explain why dividing a palindromic polynomial of degree $2m$ by x^m turns the equation $P(x) = 0$ into an equation of degree m in u. Suggest how the investigation could continue, for example with palindromic equations of degree 6 or with polynomials whose coefficients reverse with a change of sign. [3]

32. Splitting a sum with partial fractions. Partial fractions are usually a step towards integration. This investigation uses them for something else: adding infinitely many terms exactly.

(a) Show that $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$ for every positive integer n. [2]

(b) Hence write out the sum $\displaystyle\sum_{n=1}^{N} \frac{1}{n(n+1)}$ term by term, and show that almost every term cancels. Deduce that the sum equals $\dfrac{N}{N+1}$. *(Hint: write the first three and the last two terms)* [4]

(c) State the value the sum approaches as $N \to \infty$, and explain what that value means for the graph of the partial sums. [2]

(d) Show that $\dfrac{1}{n(n+2)} = \dfrac{1}{2}\left(\dfrac{1}{n} - \dfrac{1}{n+2}\right)$, and hence find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$. [4]

(e) Investigate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+k)}$ for a positive integer k. Find its value for $k = 3$ and $k = 4$, then state a general formula and justify it. [6]

(f) Show that $\dfrac{1}{n(n+1)(n+2)} = \dfrac{1}{2}\left(\dfrac{1}{n(n+1)} - \dfrac{1}{(n+1)(n+2)}\right)$, and hence find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$. Then use $\dfrac{1}{n^2} < \dfrac{1}{n(n-1)}$ for $n \geq 2$ to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$, and suggest how the exact value of this sum could be investigated. [4]

33. Dividing by a product of two linear factors. End-of-Chapter Questions 5 and 26 find a remainder on division by $(x-a)(x-b)$ from two given values. This investigation finds that remainder in general, applies it to x^n, and then asks what happens when the two factors are the same. Throughout, $P(x)$ is a polynomial and a and b are real constants.

(a) Explain why the remainder when $P(x)$ is divided by $(x-a)(x-b)$ has the form $px + q$. Show that, when $a \neq b$, the remainder is

$$R(x) = P(a)\,\frac{x-b}{a-b} + P(b)\,\frac{x-a}{b-a}.$$

[4]

(b) Hence show that, for every positive integer n, the remainder when x^n is divided by $x^2 - 3x + 2$ is $(2^n - 1)x + 2 - 2^n$. Find also the remainder when x^n is divided by $x^2 - 1$, treating even and odd n separately. [5]

(c) The formula in part (a) cannot be used when $a = b$. By long division or otherwise, find the remainder when x^n is divided by $(x-1)^2$ for $n = 2$, 3 and 4, and conjecture the remainder for every positive integer n. [3]

(d) Prove your conjecture. *(Hint: $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$)* [5]

(e) Compare the line $y = nx + 1 - n$ with the graph of $y = x^n$ near $x = 1$, and suggest what the remainder on division by $(x-a)^2$ tells you about the graph of any polynomial P. Suggest how the investigation could be extended to division by $(x-a)(x-b)(x-c)$. [3]