Chapter 5

Functions & Graphs

5

SYLLABUS 2.5 2.8–2.11 2.13–2.16

Hold a card in front of a lamp and look at its shadow on the wall. Slide the card and the shadow slides. Move it closer to the lamp and the shadow grows. Turn it over and the shadow flips. The card itself never changes. Only what you do to it changes.

Graphs work the same way, and that makes this part of the course more manageable. There are only a few shapes to learn by heart, a line, a parabola, a square root, a reciprocal, together with a short list of moves that slide them, stretch them and reflect them. Many other curves you will meet are one of those few shapes with some of those moves applied to it.

So this chapter has two halves. The first is **composition** and the list of moves: how the output of one function becomes the input of the next, and what each move does to an equation. The second is the small set of shapes those moves are applied to, and in particular the ones whose graphs approach a line, called an asymptote. Reading that behaviour from a formula is the second skill this chapter builds.

By the end of this chapter you will be able to compose two functions and find the domain of the result, apply translations, stretches and reflections, and test a function for symmetry. You will sketch the related graphs $|f(x)|$, $f(|x|)$ and $\dfrac{1}{f(x)}$. You will find the asymptotes of a rational function, solve a rational inequality with a sign table, solve equations and inequalities involving $|f(x)|$, and treat the exponential and the logarithm as inverses of each other.

What ties these together is that each constant in these formulas corresponds to something you can see. Change the number and the graph moves in a way you can predict. A formula and a picture are two views of one function, and this chapter is about reading each from the other.

5.1 Composite Functions

To work out $\sqrt{3x+1}$ you do three things, in order. Multiply by 3. Add 1. Take the square root. Many formulas in this course are built like that, from a few simple steps joined end to end. Reading a formula as a chain of steps is what lets you undo it, and later what lets you differentiate it. **Composition** is the notation for that chain.

Two functions can be joined end to end. Send a number into the first function. Take the number that comes out. Send that number into the second function. The single rule that does both steps at once is called a **composite** function.

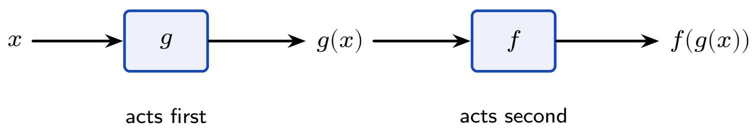


Figure 5.1 Composition sends x through g first, then sends the result into f. The inner function acts first, which is why $f \circ g$ is read from the inside out.

> DEF The **composite** $f \circ g$ is the function $(f \circ g)(x) = f(g(x))$: apply g first, then f. The **identity function** $I(x) = x$ returns every input unchanged.

The symbol $\circ$ is read "of", so $f \circ g$ is "f of g". It is also read "f after g", which gives the order in which the two functions act. It is not multiplication. Read the notation from right to left: the function written closest to x acts on x first.

5.1.1 Substituting a Whole Function

Composition looks unfamiliar because a function is being written where a number normally goes. The rule for f does not change at all. Only what you write in the slot changes.

The x in $f(x)$ is a **placeholder** (a marker showing where the input goes). It is not a fixed value, and it does not have to be a single letter.

Read f as a rule with an empty slot. If $f(x) = x^2 + 1$, the rule says: square the input, then add one. Whatever you write in the slot is squared, and then one is added.

$$f(\square) = \square^2 + 1 \qquad \Longrightarrow \qquad f(3) = 3^2 + 1, \quad f(2x) = (2x)^2 + 1, \quad f(g(x)) = \big(g(x)\big)^2 + 1.$$

The last one is composition. You are substituting a whole function exactly as you would substitute a number.

To build $(f \circ g)(x)$:

1. Write the formula for $f(x)$.
2. Substitute the expression $g(x)$ for **every** x in that formula.
3. Simplify.

5.1.2 Order of Composition

Putting on socks and then shoes is not the same as putting on shoes and then socks. Composition behaves in the same way, so $f \circ g$ and $g \circ f$ are usually two different functions. Deciding which function acts first is therefore a good first step in a composition question.

Worked through. Take $f(x) = x^2 + 1$ and $g(x) = 2x$. For $f \circ g$, apply g first. Its output is $2x$, so substitute $2x$ for every x in $f(x) = x^2 + 1$, then simplify:

$$(f \circ g)(x) = f(2x) = (2x)^2 + 1 = 4x^2 + 1.$$

For $g \circ f$, apply f first. Its output is $x^2 + 1$, so substitute $x^2 + 1$ for every x in $g(x) = 2x$, then simplify:

$$(g \circ f)(x) = g(x^2 + 1) = 2(x^2 + 1) = 2x^2 + 2.$$

The two answers are different. In general $f \circ g \neq g \circ f$, so always check which function acts first.

One input makes the difference concrete. Send $x = 3$ through both composites and follow it step by step.

Composite	Input	After the first function	After the second function
$f \circ g$	3	$g(3) = 6$	$f(6) = 37$
$g \circ f$	3	$f(3) = 10$	$g(10) = 20$

5.1.3 The Domain of a Composite

Each part of a composite can be defined at a number while the composite is not. The reason is that f is never applied to x. It is applied to $g(x)$, and $g(x)$ may be a value that f does not take. So two conditions must hold, and the second one is easy to miss.

KEY The **domain** of $f \circ g$ is the set of values x for which

1. x is in the domain of g, **and**
2. $g(x)$ is in the domain of f.

Find the domain from the two original functions, before simplifying. Simplification is an algebraic step, and a simplified formula may be defined at values where the composite is not.

Worked through. To find $(f \circ g)(x)$ and its domain for $f(x) = x^2$ and $g(x) = \sqrt{x-1}$, apply g first, then square its output:

$$(f \circ g)(x) = f(\sqrt{x-1}) = \left(\sqrt{x-1}\right)^2 = x - 1.$$

Now take the two conditions in order.

Condition 1: the domain of g requires $x - 1 \geq 0$, so $x \geq 1$.

Condition 2: the domain of f is all real numbers, so this adds nothing here.

The domain of $f \circ g$ is therefore $x \geq 1$. The composite is the line $y = x - 1$ with everything to the left of $x = 1$ removed.

CAUTION A simplified formula is not the composite. $\left(\sqrt{x-1}\right)^2$ simplifies to $x - 1$, which is defined for every real x, but the composite exists only for $x \geq 1$. Always return to the original functions to find the domain.

EXAMPLE 5.1

For $f(x) = \dfrac{1}{x}$ and $g(x) = x - 3$, find $(f \circ g)(x)$ and state its domain.

$$(f \circ g)(x) = f(x-3) = \frac{1}{x-3}.$$

Condition 1: the domain of g is all real numbers, so it removes nothing.
Condition 2: 0 is not in the domain of f, so the output $g(x) = x - 3$ must not equal 0. This removes $x = 3$.
The domain of $f \circ g$ is every real number except $x = 3$.

The two cases above restrict the domain for different reasons, which is why both conditions have to be checked every time.

The functions	1. x in the domain of g	2. $g(x)$ in the domain of f	Domain of $f \circ g$
$f(x) = x^2$ $g(x) = \sqrt{x-1}$	$x \geq 1$	no extra condition	$x \geq 1$
$f(x) = \dfrac{1}{x}$ $g(x) = x - 3$	every real x	$x - 3 \neq 0$, so $x \neq 3$	$x \in \mathbb{R}$, $x \neq 3$

5.1.4 Composition and Inverses

Some pairs of functions undo each other. Composition is how that relationship is written down, and composing the two is how you check that an inverse is correct.

The **inverse** of f, written f^{-1}, is the function that reverses f. If f sends 3 to 10, then f^{-1} sends 10 back to 3. Apply f and then apply f^{-1}, and you arrive back at the number you started with. The composite is therefore the identity function.

KEY For a function f with inverse f^{-1},

$$(f^{-1} \circ f)(x) = x \qquad \text{and} \qquad (f \circ f^{-1})(x) = x.$$

An inverse exists exactly when f is **one-to-one**: no two inputs share an output.

To find an inverse, use the method of Ch. 4, where inverses were first met:

1. Write $y = f(x)$.
2. Exchange x and y.
3. Solve the new equation for y. The result is $f^{-1}(x)$.
4. State the domain of f^{-1}, which is the range of f.

Composition is also how you check the answer. Take $f(x) = 2x + 1$, whose inverse is $f^{-1}(x) = \dfrac{x-1}{2}$. Substituting one into the other gives

$$f^{-1}(f(x)) = \frac{(2x+1)-1}{2} = \frac{2x}{2} = x,$$

which confirms the pair. An important pair of inverse functions in this chapter is the exponential and the logarithm; see §5.5.

A quotient of two linear expressions needs one extra move at step 3. The variable appears twice, once above the line and once below it, so it has to be collected into a single term before it can be isolated. The range of such a function, which is the domain of its inverse, is worked out in §5.4.1.

EXAMPLE 5.2

Find the inverse of $f(x) = \dfrac{3x-1}{x-1}$, and state its domain.
Write $y = f(x)$ and exchange the letters:

$$x = \frac{3y-1}{y-1}.$$

Clear the fraction by multiplying both sides by $y - 1$:

$$x(y-1) = 3y - 1 \implies xy - x = 3y - 1.$$

Now y is in two terms. Gather those on one side and everything else on the other, then factorise and divide:

$$xy - 3y = x - 1 \implies y(x-3) = x - 1 \implies f^{-1}(x) = \frac{x-1}{x-3}, \qquad x \neq 3.$$

The excluded value is not guesswork. The domain of f^{-1} is the range of f, and f never produces 3: written as $f(x) = 3 + \dfrac{2}{x-1}$, its second term is never zero, so the output is never exactly 3.

Check one point: $f(0) = \dfrac{-1}{-1} = 1$, and $f^{-1}(1) = \dfrac{0}{-2} = 0$, which returns the input.

YOUR TURN 5.1 Composite Functions

Forming a composite

1. For $f(x) = x + 3$ and $g(x) = 2x$, find each composite.

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$
(c) $(f \circ f)(x)$ (d) $(g \circ g)(x)$

2. For $f(x) = x^2$ and $g(x) = x - 1$, find each of the following.

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$
(c) $(f \circ g)(3)$ (d) $(g \circ f)(3)$

3. Let $f(x) = x^2$ and $g(x) = x + 2$.

(a) Find $(f \circ g)(x)$ and $(g \circ f)(x)$, and show that they are not the same.
(b) Find the value of x for which the two composites are equal.

The domain of a composite

4. For $f(x) = \dfrac{1}{x}$ and $g(x) = x + 1$, find each composite and state its domain.

(a) $(f \circ g)(x)$ (b) the domain of $f \circ g$
(c) $(g \circ f)(x)$ (d) the domain of $g \circ f$

5. For each pair of functions, find $(f \circ g)(x)$ and state the domain of $f \circ g$. Find the domain from f and g before you simplify.

(a) $f(x) = \sqrt{x}, \quad g(x) = 3 - x$
(b) $f(x) = x^2, \quad g(x) = \sqrt{x+2}$
(c) $f(x) = \dfrac{1}{x-2}, \quad g(x) = x^2 - 2$
(d) $f(x) = \sqrt{x-1}, \quad g(x) = x^2$

Composites and inverses

6. Find the inverse of each function, and state the domain of the inverse.

(a) $f(x) = \dfrac{x+3}{x-2}$ (b) $f(x) = \dfrac{3x}{2x-1}$

7. Let $f(x) = 2x + 1$.

(a) Find $f^{-1}(x)$.
(b) Show that $(f \circ f^{-1})(x) = x$.
(c) Show that $(f^{-1} \circ f)(x) = x$.
(d) State what these two results tell you about f^{-1}.

Concept check

8. Let $f(x) = x + 3$ and $g(x) = x^2$. A student writes: "$(f \circ g)(x) = (x+3)^2$." Identify the error and find $(f \circ g)(x)$.

9. Let $f(x) = x^2$ and $g(x) = \sqrt{x+2}$. A student writes: "$(f \circ g)(x) = \left(\sqrt{x+2}\right)^2 = x+2$, *so the domain of* $f \circ g$ *is* $x \in \mathbb{R}$." Identify the error and state the correct domain.

5.2 Transformations of Graphs

Plotting points is slow, and for most curves it is unnecessary. A small change to an equation moves the graph in one predictable way. Once you know which change produces which move, you can sketch a new curve from a familiar one in a few seconds. You can also work the other way, and write down the equation of a curve you are shown.

There are three moves: a **translation** (a slide), a **reflection** (a flip) and a **stretch** (a scaling). Each one matches one change in the equation, and each one sends the point (p, q) to a new point you can write down.

Move	Equation	Acts on	(p,q) becomes	Effect
Translation	$y = f(x) + k$	output	$(p,\ q+k)$	k up
	$y = f(x-h)$	input	$(p+h,\ q)$	h right
Reflection	$y = -f(x)$	output	$(p,\ -q)$	in the x-axis
	$y = f(-x)$	input	$(-p,\ q)$	in the y-axis
Stretch	$y = a\,f(x)$	output	$(p,\ aq)$	vertical, factor a
	$y = f(bx)$	input	$(\frac{p}{b},\ q)$	horizontal, factor $\frac{1}{b}$

The table is three pairs, one pair for each move. Read the **Acts on** column and a single pattern accounts for all six rows. Everything depends on whether the change sits outside the bracket or inside it.

> KEY **Outside** the bracket, the change acts on the output: it is **vertical**, and it does what it says. **Inside** the bracket, the change acts on the input: it is **horizontal**, and it does the **opposite**.

The first line behaves as you would expect. Adding k raises every output by k, so the graph moves up by k. Multiplying by a multiplies every output by a, so the graph stretches vertically by a.

The second line needs an explanation, because the reversal is not obvious. Take $y = f(x-h)$ and evaluate it at $x = p+h$:

$$f((p+h)-h) = f(p),$$

which is the value the original graph had at $x = p$. So the point that was at $x = p$ now appears at $x = p+h$. Every point moves h units to the *right*, not left. The same calculation works for $y = f(bx)$: at $x = \frac{p}{b}$ it returns $f(b \cdot \frac{p}{b}) = f(p)$, so the point at p moves to $\frac{p}{b}$. That is a horizontal stretch of factor $\frac{1}{b}$, which brings every point closer to the y-axis whenever $b > 1$.

5.2.1 Translations

A translation slides the whole graph. Its shape and its size do not change, only its position.

Vertically, $y = f(x) + k$ adds k to every output, so every point moves up by k. A negative k moves the graph down.

Horizontally, $y = f(x-h)$ moves the graph right by h. This sign is easy to reverse. Use the reason above rather than memorising the result. The change is inside the bracket, so it acts on the input, and its effect is the opposite of its sign.

In examination questions a translation is usually written as a vector. The graph of $y = f(x-h)+k$ is the graph of $y = f(x)$ translated by $\binom{h}{k}$.

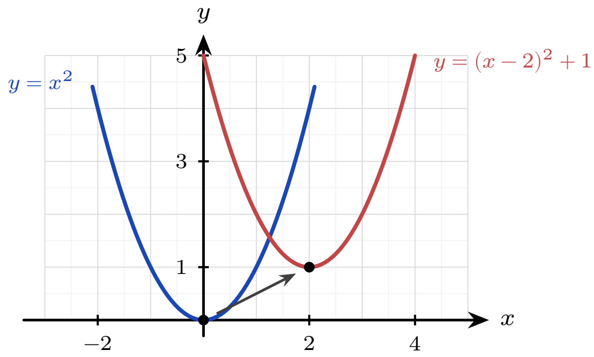


Figure 5.2 The graph of $y = (x-2)^2 + 1$ is the graph of $y = x^2$ translated 2 units right and 1 unit up. The vertex moves from $(0, 0)$ to $(2, 1)$, and the shape is unchanged.

Worked through. To obtain $y = (x-2)^2 + 1$ from $y = x^2$, start with the bracket. The -2 sits inside it, so it acts on the input and moves the parabola 2 units *right*. Now take the rest. The $+1$ sits outside the bracket, so it acts on the output and moves the parabola 1 unit *up*. Together these send the vertex from $(0, 0)$ to $(2, 1)$, which is exactly what the completed-square form predicts. Transformations and the completed-square form are two descriptions of the same thing.

5.2.2 Reflections

A reflection flips the graph across an axis. Every point keeps its distance from that axis and moves to the other side of it.

For $y = -f(x)$ the sign of every output changes, so the point (p, q) moves to $(p, -q)$. That is a reflection in the x-axis. A maximum becomes a minimum, and any point already on the x-axis stays where it is.

For $y = f(-x)$ the sign of every input changes, so the point (p, q) moves to $(-p, q)$. That is a reflection in the y-axis. The y-intercept does not move, because changing the sign of 0 leaves it as 0.

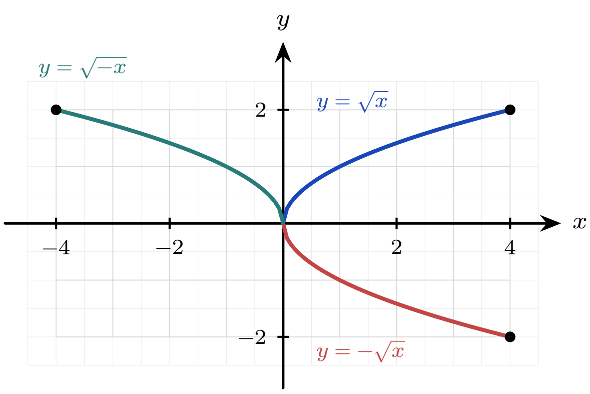


Figure 5.3 Reflections of $y = \sqrt{x}$. Changing the sign of the output gives $y = -\sqrt{x}$, and the point $(4, 2)$

moves to $(4, -2)$: a reflection in the x-axis. Changing the sign of the input gives $y = \sqrt{-x}$, and $(4, 2)$ moves to $(-4, 2)$: a reflection in the y-axis.

EXAMPLE 5.3

Describe how the graph of $y = -(x-1)^2$ is obtained from $y = x^2$.

There are two changes. The -1 is inside the bracket, so it is horizontal: translate 1 unit right.

The minus sign is outside the bracket, so it is vertical: reflect in the x-axis.

One change is horizontal and the other is vertical, so they do not affect each other and may be done in either order.

The vertex moves from $(0, 0)$ to $(1, 0)$, and the parabola now opens downward.

5.2.3 Stretches

A stretch changes the size of the graph in one direction only. One line stays fixed while every other point moves away from it or towards it.

For $y = a\,f(x)$ every output is multiplied by a, so (p, q) moves to (p, aq). This is a vertical stretch of factor a. Points on the x-axis do not move, because $a \times 0 = 0$.

For $y = f(bx)$ every input is divided by b, as shown above, so (p, q) moves to $(\frac{p}{b}, q)$. This is a horizontal stretch of factor $\frac{1}{b}$. Points on the y-axis do not move.

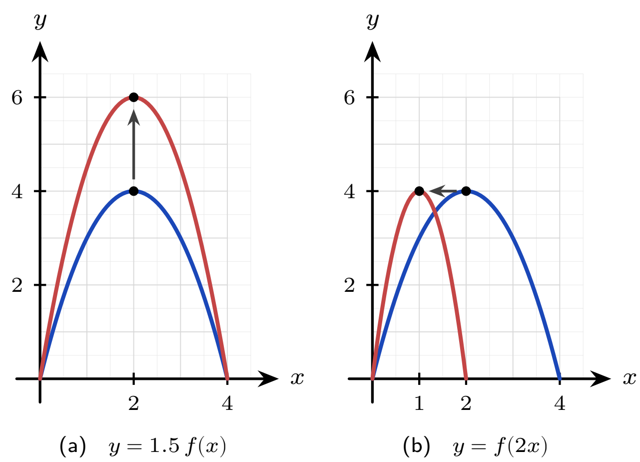


Figure 5.4 The original curve $y = f(x)$ is cobalt and the stretched curve is scarlet. In (a) every y-coordinate is multiplied by 1.5, so the maximum moves from $(2, 4)$ to $(2, 6)$ while both roots stay on the x-axis. In (b) every x-coordinate is halved, so the maximum moves from $(2, 4)$ to $(1, 4)$ and the root at $x = 4$ moves to $x = 2$.

CAUTION A factor smaller than 1 is still called a stretch. The graph of $y = \frac{1}{2}f(x)$ is a vertical stretch of factor $\frac{1}{2}$, even though it makes the graph shorter. Write "stretch, factor $\frac{1}{2}$", not "squash".

EXAMPLE 5.4

The graph of $y = f(x)$ has a maximum point at $(6, 4)$. State the coordinates of that maximum point on the graph of (a) $y = 3f(x)$, (b) $y = f(3x)$.

(a) The 3 is outside the bracket, so it is a vertical stretch of factor 3. It multiplies the y-coordinate and leaves the x-coordinate alone: $(6, 12)$.

(b) The 3 is inside the bracket, so it is a horizontal stretch of factor $\frac{1}{3}$. It divides the x-coordinate and leaves the y-coordinate alone: $(2, 4)$.

Each stretch changes one coordinate and leaves the other unchanged. Checking which coordinate moved is a quick way to confirm you have used the right one.

5.2.4 Composite Transformations

Many graphs in this chapter are several moves away from a familiar shape, applied one after another. This is a **composite transformation**. Describing one is a matter of taking the changes in a reliable order.

Start by writing the target equation in one standard form:

$$y = a\,f(b(x - h)) + k.$$

Then a is the vertical stretch factor, b gives the horizontal stretch factor $\frac{1}{b}$, h is the horizontal translation and k is the vertical translation. Every value can be read straight from the equation.

For a quadratic this standard form is the **completed-square form**. Writing $y = a(x - h)^2 + k$ names the vertex (h, k) and the two moves that put it there, so completing the square and describing a transformation are two ways of doing the same work.

KEY To describe a composite transformation:

1. Split the changes into two groups: those **inside** the bracket, which are horizontal, and those **outside** it, which are vertical.
2. Do the two groups in either order. A horizontal move and a vertical move do not affect each other.
3. Inside each group the order matters. Apply the **stretch before the translation**.

Step 3 has a reason, and understanding it is safer than memorising it. A stretch multiplies every value it is applied to, including a translation that has already been carried out. Do the stretch first and there is no translation for it to multiply. The example at the end of this subsection shows what goes wrong in the other order.

Whatever sequence you write down, check it by tracking a single point. Choose a point you know on the original graph, such as a vertex, an endpoint or an intercept. Apply your moves to its coordinates. Then check that the result satisfies the target equation.

CAUTION Factorise inside the bracket first. In $y = f(2x - 6)$ the translation is *not* 6. Written as $y = f(2(x - 3))$, it is a horizontal stretch of factor $\frac{1}{2}$ followed by a translation of 3 to the right. Forgetting to factorise is a common error here.

Worked through. To describe a sequence of transformations that maps $y = \sqrt{x}$ onto $y = 3\sqrt{x-2} + 1$, take the change inside the bracket first. The -2 acts on the input, so it translates the graph 2 units *right* and gives $\sqrt{x-2}$.

Now take the vertical changes, stretch before translation. The factor 3 stretches the graph vertically by scale factor 3 and gives $3\sqrt{x-2}$. The $+1$ then translates it 1 unit *up*.

So the sequence is: translate 2 right, stretch vertically by factor 3, translate 1 up.

Check it by tracking the endpoint. The graph of $y = \sqrt{x}$ begins at $(0, 0)$. The translation carries it to $(2, 0)$. The stretch leaves it there, because $3 \times 0 = 0$. The final translation carries it to $(2, 1)$. That is exactly where $y = 3\sqrt{x-2} + 1$ begins.

Two moves **commute** (give the same result in either order) when swapping them leaves the final graph unchanged. A horizontal move and a vertical move always commute, because they change different coordinates. A stretch and a translation in the *same* direction do not commute, unless the stretch factor is 1 or the translation is 0, so that one of the two moves does nothing. The next example shows the two different answers they produce.

EXAMPLE 5.5

Starting from $y = x^2$, apply a vertical stretch by factor 2 and a translation 1 unit up, in both orders. Are the results the same?

Stretch first, then translate. The stretch doubles every value, giving $2x^2$; the translation then adds 1:

$$x^2 \longrightarrow 2x^2 \longrightarrow 2x^2 + 1.$$

Translate first, then stretch. The translation adds 1, giving $x^2 + 1$; the stretch then doubles everything, *including* the 1:

$$x^2 \longrightarrow x^2 + 1 \longrightarrow 2(x^2 + 1) = 2x^2 + 2.$$

The two curves are different. A vertical stretch multiplies whatever is already there, so a translation applied first is multiplied as well. A vertical stretch and a vertical translation commute only when one of them does nothing, that is when the stretch factor is 1 or the translation is 0. So state the moves in the order that builds the target curve.

TIP In examinations, "describe the sequence of transformations" expects named moves with their values: translation by $\binom{-3}{0}$, vertical stretch scale factor 2, translation by $\binom{0}{-1}$. Vague words such as "moved" or "squashed" do not earn the marks.

YOUR TURN 5.2 Transformations of Graphs

Single transformations

10. Describe the single transformation that takes $y = x^2$ to each graph.

(a) $y = x^2 + 3$ (b) $y = (x+4)^2$
(c) $y = -x^2$ (d) $y = (x-1)^2$

11. Describe the single transformation that takes $y = f(x)$ to each graph.

(a) $y = f(-x)$ (b) $y = -f(x)$
(c) $y = 3f(x)$ (d) $y = f(2x)$

The image of a point

12. The point $(2, 5)$ lies on $y = f(x)$. Find the coordinates of its image on each graph.

(a) $y = f(x-3) + 1$ (b) $y = 3f(x)$
(c) $y = f(2x)$ (d) $y = -f(x) + 2$

13. The point $(-2, 3)$ lies on $y = f(x)$. Find the coordinates of its image on each graph.

(a) $y = f(2x)$ (b) $y = f(x) - 4$
(c) $y = f(-x)$ (d) $y = \frac{1}{2}f(x)$

Combined transformations

14. Describe the sequence of transformations that maps $y = \sqrt{x}$ onto each graph.

(a) $y = \sqrt{x+3}$ (b) $y = 2\sqrt{x}$
(c) $y = 2\sqrt{x+3} - 1$ (d) $y = \sqrt{3x+6}$ *(Hint: factorise inside the root first)*

15. Start from $y = x^2$.

(a) Translate 2 units up, then reflect in the x-axis. Write the new equation.
(b) Apply the same two moves in the opposite order. Write the new equation.

16. Consider $y = f(4x - 8)$.

(a) Write it in the form $y = f(b(x-h))$.
(b) Describe the sequence of transformations from $y = f(x)$.

Concept check

17. A student describes two transformations of $y = f(x)$. Identify the error in each description and give the correct one.

(a) "$y = f(x+3)$ *is a translation of* 3 *units to the right.*"

(b) "$y = f(2x)$ *is a horizontal stretch with scale factor* 2."

5.3 Symmetry, Inverses and Related Graphs HL

Half of a sketch is half the work, and for some graphs half is all you need. A graph has a **symmetry** when some motion leaves it looking exactly as it did. Reflect it, rotate it or slide it, and the picture is unchanged. One half of the curve then fixes the other half, and the symmetry gives you a way to check a sketch against the formula.

Three motions are common enough to be named, and each one has a short algebraic test.

Type	Test	Motion that leaves the graph unchanged	Examples
Even	$f(-x) = f(x)$	reflection in the y-axis	x^2, x^4, $\lvert x\rvert$, $\cos x$
Odd	$f(-x) = -f(x)$	rotation of 180° about the origin	x, x^3, $\frac{1}{x}$, $\sin x$
Periodic	$f(x+p) = f(x)$	translation of p along the x-axis	$\sin x$, $\cos x$, $\tan x$
Neither	no test holds	none of the three	$x^2 + x$, $x+1$, e^x

Most functions have none of these symmetries, so "neither" is a complete and correct answer when the tests fail.

The third row is the one you meet again in trigonometry. A function is **periodic** if its graph repeats at regular intervals, and the smallest positive p with $f(x+p) = f(x)$ is called the **period**. The graphs of $\sin x$ and $\cos x$ both have period 2π.

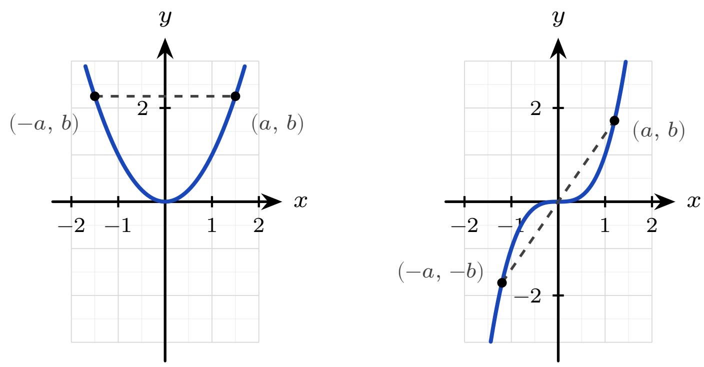
5.3.1 Testing for Even and Odd

The names come from the powers. The functions $x^2, x^4, \dots$ have even exponents and satisfy $f(-x) = f(x)$. The functions $x, x^3, \dots$ have odd exponents and satisfy $f(-x) = -f(x)$.

DEF A function is **even** if $f(-x) = f(x)$ for every x in its domain, and **odd** if $f(-x) = -f(x)$ for every x in its domain. An even graph is symmetric in the y-axis; an odd graph has rotational symmetry of order two about the origin.

To test a function, substitute $-x$ for x, simplify, and compare the result with the original. There are three possible outcomes:

- the result is $f(x)$, so the function is **even**;
- the result is $-f(x)$, so the function is **odd**;
- the result is neither, so the function has **no symmetry** of this kind.



(a) $y = x^2$ is even (b) $y = x^3$ is odd

Figure 5.5 In (a) the point (a, b) has a partner at $(-a, b)$, directly across the y-axis, so reflecting in that axis leaves the curve unchanged. In (b) the partner is at $(-a, -b)$, directly opposite through the origin, so a half turn about the origin leaves the curve unchanged.

Worked through. To decide whether $f(x) = x^3 - x$ is odd, even or neither, substitute $-x$ for x, simplify, then look for a factor of -1:

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x).$$

The result is $-f(x)$, so by the definition the function is **odd**. Its graph therefore has rotational symmetry of order two about the origin.

5.3.2 Self-Inverse Functions

A fourth symmetry belongs to functions that undo themselves, and it is usually met as an algebraic expression rather than as a graph. Recall that f^{-1} reverses f. If applying f twice already returns the input, then f reverses itself.

DEF A function is **self-inverse** if $f(f(x)) = x$ for every x in its domain. Then $f = f^{-1}$, and the graph of f is symmetric in the line $y = x$.

That symmetry has a simple reason. Reflecting any graph in the line $y = x$ produces the graph of its inverse, because the reflection swaps the two coordinates of every point. If f is its own inverse, the reflection has to return the same curve.

To test for self-inverse, substitute the whole expression for $f(x)$ in place of x, simplify, and check whether the result is x.

Worked through. To show that $f(x) = \dfrac{1}{x}$ is self-inverse, apply the function twice. Substitute the whole expression $\frac{1}{x}$ in place of x:

$$f(f(x)) = \frac{1}{\frac{1}{x}} = x.$$

Applying f twice returns the input, so f reverses itself and $f = f^{-1}$. Its graph must therefore be symmetric in the line $y = x$, which the figure below confirms.

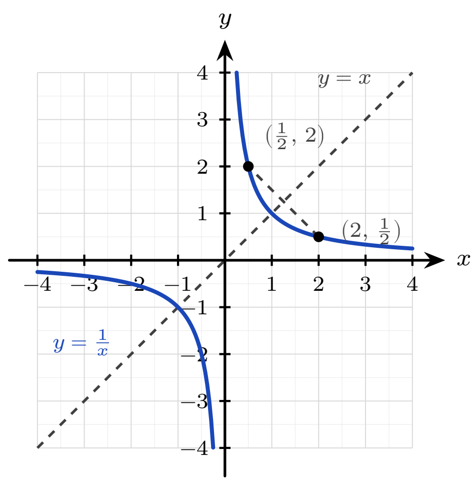


Figure 5.6 The reciprocal is symmetric in the line $y = x$. Each point on the curve has a partner with its two coordinates swapped, so reflecting in $y = x$ returns the same curve.

Self-inverse functions are not rare, and they are not all this simple. The reflection $f(x) = 6 - x$ is self-inverse, and so is $f(x) = \dfrac{2x+1}{x-2}$, which you can check by the same substitution.

5.3.3 Restricting the Domain to Get an Inverse

An inverse exists only when no two inputs share an output. Many familiar functions fail that test on their full domain, and a smaller domain makes them one-to-one.

For example, $f(x) = x^2$ is not one-to-one on all of $\mathbb{R}$, since $f(2) = f(-2)$, so it has no inverse there. Restricting the domain to $x \geq 0$ makes it one-to-one, and the inverse is $f^{-1}(x) = \sqrt{x}$.

Restricting a domain is a standard examination technique, and the restriction you choose decides which branch of the inverse you get. Two facts hold for any inverse. The domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

Worked through. To find $f^{-1}(x)$ for $f(x) = (x-3)^2$, $x \geq 3$, and state its domain, first check that an inverse exists. On $x \geq 3$ the parabola only rises, so no two inputs share an output and f is one-to-one.

Now find the inverse. Write $y = f(x)$, exchange the letters, and solve for y:

$$x = (y-3)^2 \implies y - 3 = \pm\sqrt{x} \implies y = 3 \pm \sqrt{x}.$$

Two branches appear, so the restriction must decide between them. The range of f^{-1} is the domain of f, so $y \geq 3$. Then $y - 3$ cannot be negative, and only the $+$ branch is possible. Hence

$$f^{-1}(x) = 3 + \sqrt{x}, \qquad x \geq 0,$$

where the domain of f^{-1} is the range of f. Had the restriction been $x \leq 3$ instead, the same algebra would give $f^{-1}(x) = 3 - \sqrt{x}$. The algebra is the same until that final choice of sign, but the inverse is different.

5.3.4 The Modulus Graph $|f(x)|$

The next two graphs, $y = |f(x)|$ and $y = f(|x|)$, are also produced by reflections. None of the remaining graphs in this section needs a table of values. Each is built from a function f by one operation, and each has a single visual rule.

The modulus of a number is its size without its sign, so $|f(x)|$ is never negative. That gives the rule at once.

> KEY To sketch $y = |f(x)|$: keep every part of $y = f(x)$ that lies on or above the x-axis, and reflect every part below the axis upward in the x-axis. The x-intercepts do not move, because $|0| = 0$. A root where f *crosses* the axis (where the gradient of f is not zero) becomes a corner; a root where f merely touches the axis, as in $y = x^2$, stays smooth.

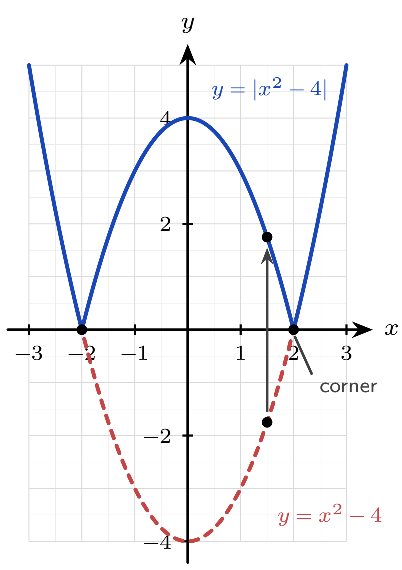


Figure 5.7 $y = |x^2 - 4|$ against $y = x^2 - 4$. The part of the parabola below the x-axis is reflected upward, shown by the arrow, which carries the point $(1.5, -1.75)$ to $(1.5, 1.75)$; everything already above the axis

is untouched. The two roots stay at $x = \pm 2$ and become corners.

EXAMPLE 5.6

Sketch $y = |f(x)|$ for $f(x) = x^2 - 6x + 8$, stating the coordinates of the intercepts, the corners and the turning point.

Find the key points of f first. Factorising gives $f(x) = (x-2)(x-4)$, so the roots are $x = 2$ and $x = 4$, and the vertex lies halfway between them at $x = 3$, where $f(3) = 9 - 18 + 8 = -1$. So f has roots $(2, 0)$ and $(4, 0)$ and a minimum at $(3, -1)$.

Only the arc between the roots lies below the x-axis, so only that arc is reflected. Everything outside $2 \leq x \leq 4$ is already above the axis and is left alone.

Now take the key points one at a time. The roots $(2, 0)$ and $(4, 0)$ do not move, because $|0| = 0$, and f *crosses* the axis at each of them, so each becomes a **corner**. The minimum $(3, -1)$ is reflected to $(3, 1)$, and it is now a local **maximum**. The y-intercept is $f(0) = 8$, which is already positive and so is left where it is, at $(0, 8)$.

So the curve falls from the upper left to the corner $(2, 0)$, rises to the peak $(3, 1)$, falls to the corner $(4, 0)$, then rises again to the upper right, and no part of it lies below the x-axis.

The peak sits at height $|-1| = 1$: reflecting in the x-axis changes the sign of the y-coordinate and leaves the x-coordinate where it was.

This shape is also what makes modulus equations solvable, and §5.3.8 uses it.

5.3.5 The Graph of $f(|x|)$

The previous rule applied the modulus to the *output*. Applying it to the *input* instead gives a different graph and a different rule, and questions on this pair often test whether you can tell them apart.

Since $|x|$ and $|-x|$ are the same number, f receives the same input at x and at $-x$, so the two outputs are equal. The graph is therefore its own mirror image in the y-axis.

KEY To sketch $y = f(|x|)$: keep the part of $y = f(x)$ with $x \geq 0$, and reflect it in the y-axis to make the part with $x < 0$. The original graph to the left of the y-axis is discarded. The result is always an even function. The two halves meet on the y-axis. Unless f has zero gradient at $x = 0$, they meet at a corner, which is a local maximum or minimum of $f(|x|)$.

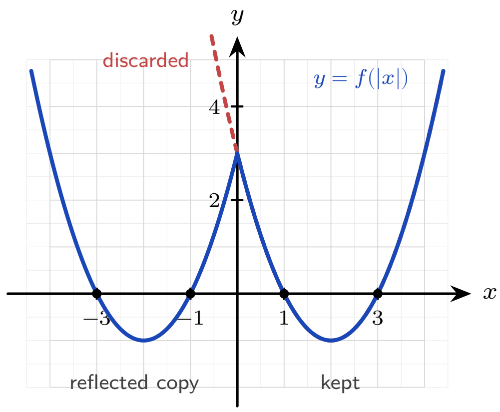


Figure 5.8 $y = f(|x|)$ for $f(x) = x^2 - 4x + 3$. The half with $x \geq 0$ is copied across the y-axis, so the roots at $x = 1$ and $x = 3$ gain partners at $x = -1$ and $x = -3$. The short dashed piece is the part of f that is discarded.

EXAMPLE 5.7

Sketch $y = f(|x|)$ for $f(x) = x^2 - 6x + 8$, stating the coordinates of the intercepts, the corner and the turning points.

Keep the half of $y = f(x)$ with $x \geq 0$ and discard the rest. That half carries the y-intercept $(0, 8)$, the roots $(2, 0)$ and $(4, 0)$, and the minimum $(3, -1)$.

Reflecting it in the y-axis copies each of those points across. The point $(0, 8)$ is on the axis and stays where it is; $(2, 0)$ gains a partner at $(-2, 0)$; $(4, 0)$ gains a partner at $(-4, 0)$; and $(3, -1)$ gains a partner at $(-3, -1)$.

So the graph meets the x-axis at $(-4, 0)$, $(-2, 0)$, $(2, 0)$ and $(4, 0)$, and it has two minima, at $(-3, -1)$ and $(3, -1)$, both of value -1. The y-intercept is $(0, 8)$.

The y-axis is where the two halves join, and they do not join smoothly. Just to the right of it the kept half is falling, with gradient -6 at $x = 0$; just to the left the mirror image is rising, with gradient $+6$. The two pieces therefore meet at a **corner** at $(0, 8)$, which is a local maximum of value 8. The result is a W shape, symmetric in the y-axis.

The formula agrees: $f(|x|) = |x|^2 - 6|x| + 8 = x^2 - 6|x| + 8$, which is unchanged when x is replaced by $-x$, so it is even, and the discarded half of f leaves no trace in it.

5.3.6 The Graph of $[f(x)]^2$

Squaring the output also removes the sign, so $[f(x)]^2$ and $|f(x)|$ share a rule: neither is ever negative. They differ in size, because squaring changes the size and a modulus does not.

Every output is squared, so this rule has two parts.

KEY To sketch $y = [f(x)]^2$: no output is negative, so the whole curve lies on or above the x-axis. Each root of f becomes a point where the new curve **touches** the axis instead of crossing it. Where $|f(x)| > 1$ the squared value is larger, so the curve rises. Where $|f(x)| < 1$ the squared value is smaller, so the curve moves closer to the axis.

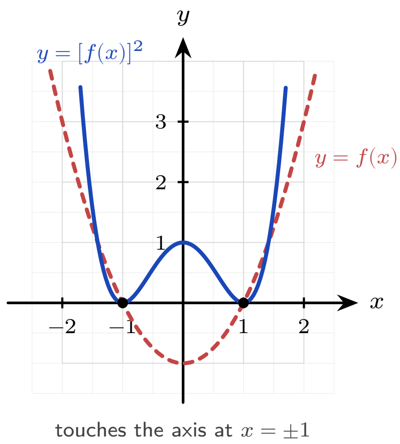


Figure 5.9 $y = [f(x)]^2$ for $f(x) = x^2 - 1$. The roots at $x = \pm 1$ become points where the squared curve touches the x-axis. For $|x| < \sqrt{2}$, $|f(x)| \le 1$, so the squared curve lies on or below $y = |f(x)|$; for $|x| > \sqrt{2}$, $|f(x)| > 1$, so it rises faster. The curves $y = |f(x)|$ and $y = [f(x)]^2$ meet where $|f(x)|$ is 0 or 1.

EXAMPLE 5.8

Sketch $y = [f(x)]^2$ for $f(x) = x^2 - 6x + 8$, stating where the curve meets the x-axis and the coordinates of its turning points.

The key points of f are again the roots $(2, 0)$ and $(4, 0)$ and the minimum $(3, -1)$. Squaring acts on the output only, so each of them keeps its x-coordinate.

Each root gives $0^2 = 0$, so the new curve still meets the x-axis at $(2, 0)$ and $(4, 0)$. It cannot cross there, because no output is negative, so at each root it **touches** the axis and turns back: both are minima, of value 0.

The minimum $(3, -1)$ gives $(-1)^2 = 1$, so the curve passes through $(3, 1)$. Between the two roots the squared value rises from 0 to 1 and falls back to 0, which makes $(3, 1)$ a local **maximum**.

Outside $2 \le x \le 4$ the size $|f(x)|$ passes 1 and keeps growing, so the squared curve climbs far faster than f does: at $x = 5$, $f(5) = 3$ while $[f(5)]^2 = 9$.

So the sketch drops steeply to touch at $(2, 0)$, arches up to $(3, 1)$, drops back to touch at $(4, 0)$, then climbs steeply away. Inside that arch $|f(x)| \le 1$, so squaring pulls the curve closer to the axis than $|f(x)|$ lies, and on $2 \le x \le 4$ the two curves coincide only where $|f(x)|$ is 0 or 1, at $x = 2$, $x = 3$ and $x = 4$. Beyond the arch they meet once more on each side, wherever $|f(x)| = 1$ again: solving $x^2 - 6x + 8 = 1$ gives $x = 3 \pm \sqrt{2}$, one point below $x = 2$ and one above $x = 4$. Squaring and taking a modulus agree exactly at height 0 and height 1, and nowhere else.

5.3.7 The Graph of $f(ax+b)$

The last of these graphs uses the horizontal rules of §5.2.4. Everything in $f(ax+b)$ happens inside the bracket, so every change is horizontal. Factorise the bracket first, so that the stretch and the translation can be read off separately.

> KEY $f(ax+b) = f\left(a\left(x+\frac{b}{a}\right)\right)$: a horizontal stretch of scale factor $\frac{1}{a}$, then a horizontal translation of $-\frac{b}{a}$.

Tracking a few key points, as below, confirms the order.

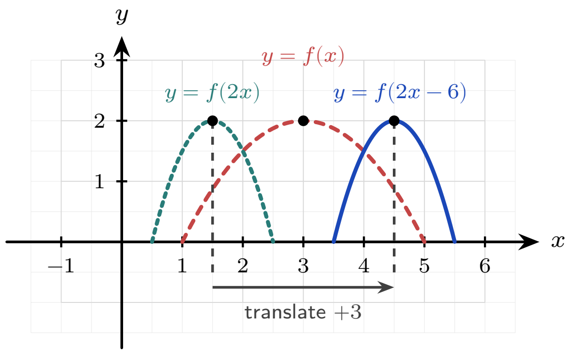


Figure 5.10 $y = f(2x-6)$, read as $f(2(x-3))$. The stretch acts first: $f(2x)$ halves the width and pulls the peak from $x=3$ to $x=1.5$. The translation of 3 to the right then carries it to $x=4.5$. Reading the -6 as a shift of 6 is a common error: the shift is $-b/a=3$, and it is applied after the stretch, not before.

EXAMPLE 5.9

Sketch $y = f(2x-6)$ for $f(x) = x^2-6x+8$, stating the images of the roots and of the vertex.
Factorise inside the bracket before reading anything off:

$$f(2x-6) = f(2(x-3)).$$

Here $a=2$ and $b=-6$, so the moves are a horizontal stretch of scale factor $\frac{1}{2}$ followed by a translation of $-\frac{b}{a}=3$ to the right. Both are horizontal, so no y-coordinate changes.
Now track the key points of f: the roots $(2,0)$ and $(4,0)$ and the vertex $(3,-1)$. The stretch halves each x-coordinate and the translation then adds 3:

$$2 \to 1 \to 4, \qquad 4 \to 2 \to 5, \qquad 3 \to \tfrac{3}{2} \to \tfrac{9}{2}.$$

So the roots move to $(4,0)$ and $(5,0)$, and the vertex moves to $(\frac{9}{2},-1)$: still a minimum, still of depth -1, and still halfway between the roots. The curve is a parabola through those three points, half as wide as f, with y-intercept $f(-6) = 36+36+8 = 80$.
Expanding confirms every number:
$f(2x-6) = ((2x-6)-2)((2x-6)-4) = (2x-8)(2x-10) = 4(x-4)(x-5)$, whose roots are 4 and 5 and whose value at $x=\frac{9}{2}$ is $4(\frac{1}{2})(-\frac{1}{2}) = -1$.

5.3.8 Modulus Equations and Inequalities

The modulus removes the sign of a number. To solve a modulus equation, consider both possible signs; what sits on the other side decides how.

There are five forms, and they must be separated before any algebra starts. Two are safe, one needs a check at the end, one is read from the V-shaped graph, and the last, where both sides contain x, may have no algebraic route at all.

A modulus equal to a number. If $|A| = k$ and k is positive, then A has size k, so it is k or $-k$. Both cases must be written down.

KEY For $k > 0$, $|A| = k$ means $A = k$ or $A = -k$.
For $k = 0$ it means $A = 0$. For $k < 0$ there is no solution, because a modulus is never negative.

Worked through. To solve $|2x - 1| = 5$, note that the right side is positive, so there are two cases:

$$2x - 1 = 5 \implies x = 3, \qquad 2x - 1 = -5 \implies x = -2.$$

So $x = 3$ or $x = -2$.

The graph agrees. The V-shaped curve $y = |2x - 1|$ meets the line $y = 5$ once on each arm.

A modulus equal to a modulus. Two sizes are equal when the numbers are equal or opposite.

KEY $|A| = |B|$ means $A = B$ or $A = -B$.
Squaring both sides gives the same two answers, because $|A|^2 = A^2$.

EXAMPLE 5.10

Solve $|2x - 1| = |x + 4|$.

$$2x - 1 = x + 4 \implies x = 5, \qquad 2x - 1 = -(x + 4) \implies 3x = -3 \implies x = -1.$$

Both work: $|9| = |9|$ and $|-3| = |3|$. So $x = 5$ or $x = -1$.

A modulus equal to an expression in x. This form looks like the first, but it needs an extra check. A modulus is never negative, so the other side cannot be negative either. Solving both cases can therefore produce a value that solves neither.

CAUTION For $|A| = B$, solve $A = B$ and $A = -B$ as usual, then **substitute each answer back** and discard any that makes B negative. These are called **extraneous solutions**. An extraneous solution can come from $A = B$ or from $A = -B$, and it is rejected because it makes B negative. For example, $|2x| = x - 3$ gives $x = -3$ from $A = B$ and $x = 1$ from $A = -B$; both make $x - 3$

negative, so the equation has no solution.

EXAMPLE 5.11

Solve $|x-1| = 2x-4$.

Two cases, as before:

$$x-1 = 2x-4 \implies x=3, \qquad x-1 = -(2x-4) \implies 3x=5 \implies x=\tfrac{5}{3}.$$

Now test each against the right side. At $x=3$: $2(3)-4=2$, which is positive, and $|3-1|=2$. That works.

At $x=\frac{5}{3}$: $2\left(\frac{5}{3}\right)-4=-\frac{2}{3}$, which is negative. A modulus cannot equal a negative number, so this value is rejected.

The only solution is $x=3$.

Inequalities. Read these from the V rather than memorising them. The graph of $y=|A|$ is a V sitting on the x-axis, and a horizontal line $y=k$ cuts it twice. Below the line is the piece between the crossings; above it are the two pieces outside.

KEY For $k>0$:

$$|A|<k \text{ means } -k<A<k, \qquad |A|>k \text{ means } A<-k \text{ or } A>k.$$

One interval for $<$, two for $>$. Use $\le$ and $\ge$ to include the endpoints.

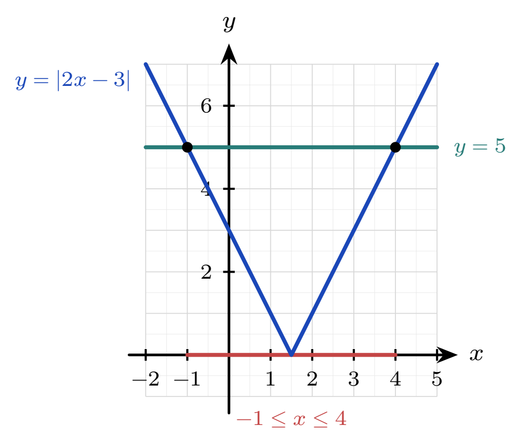


Figure 5.11 The V meets the line $y=5$ at $x=-1$ and $x=4$. Between those crossings the V is below the line, so $|2x-3|\le 5$ holds on the scarlet stretch. Outside them the V is above the line, so $|2x-3|>5$ holds there instead.

EXAMPLE 5.12

Solve (a) $|2x-3|\le 5$ and (b) $|3x+1|>7$.

(a) One interval, so write the double inequality and work on all three parts at once:

$$-5 \leq 2x - 3 \leq 5 \implies -2 \leq 2x \leq 8 \implies -1 \leq x \leq 4.$$

(b) Two intervals, so write two separate inequalities:

$$3x + 1 > 7 \implies x > 2, \qquad 3x + 1 < -7 \implies x < -\tfrac{8}{3}.$$

So $x < -\frac{8}{3}$ or $x > 2$.

The same two rules work when the expression inside is not linear, such as $|x^2 - 4| < 3$.

When both sides contain x. An inequality such as $|f(x)| > g(x)$ has no rule of its own, because the number of intervals depends on the two curves. The method is the one from §5.4.5: find where the two sides are equal, and those points cut the axis into intervals on which the answer cannot change.

KEY To solve $|f(x)| > g(x)$ or $|f(x)| < g(x)$:

1. Sketch $y = |f(x)|$ and $y = g(x)$ on the same axes.
2. Find every point where they meet, by solving $|f(x)| = g(x)$ or from a calculator.
3. Read off the intervals where the curve you want is the higher one.

TIP Many modulus inequalities have no algebraic solution at all. The syllabus example $|3x \arccos x| > 1$ is one. Graph both sides on your GDC, use intersect to find the crossing points, and read the intervals from the picture. State the intersection coordinates you used, to three significant figures.

The five forms are summarised below.

Form	Method	Watch for
$\lvert A\rvert = k$	$A = k$ or $A = -k$	no solution if $k < 0$
$\lvert A\rvert = \lvert B\rvert$	$A = B$ or $A = -B$	nothing; both roots are valid
$\lvert A\rvert = B$	two cases, then substitute back	reject any root making B negative
$\lvert A\rvert \lessgtr k$	$-k < A < k$, or two intervals	one interval for $<$, two for $>$
$\lvert f(x)\rvert \lessgtr g(x)$	find where the sides are equal, then test each interval	often needs a GDC

YOUR TURN 5.3 Symmetry, Inverses and Related Graphs HL

Odd and even functions

18. Determine whether each function is odd, even, or neither.

(a) $f(x) = x^4 - 3x^2$
(b) $f(x) = x^3 + x$
(c) $f(x) = x^2 + x$
(d) $f(x) = x^5$
(e) $f(x) = 5$
(f) $f(x) = \frac{1}{x}$

Self-inverse functions

19. A function f is self-inverse when $f(f(x)) = x$ for every x in its domain.

(a) Show that $f(x) = \frac{1}{x}$ is self-inverse.
(b) Show that $f(x) = 6 - x$ is self-inverse.
(c) Show that $f(x) = \frac{2x+1}{x-2}$ is self-inverse.
(d) Explain why $f(x) = 2x$ is not self-inverse.
(e) Explain why the graph of a self-inverse function is symmetric in the line $y = x$.
(f) Find the value of a for which $f(x) = \frac{4x+1}{2x+a}$ is self-inverse.

Restricting a domain

20. In each part, a domain is restricted so that an inverse exists.

(a) State a domain on which $f(x) = x^2$ has an inverse, and give that inverse.
(b) Do the same for $f(x) = (x-1)^2$.
(c) State the domain of the inverse you found in part (b).
(d) Explain why $f(x) = x^3$ needs none.

The graphs of $|f(x)|$, $f(|x|)$ and $[f(x)]^2$

21. The graph of $y = f(x)$ crosses the x-axis at $x = 2$, has no other root, and has a minimum point at $(0, -4)$.

(a) State where $y = |f(x)|$ meets the x-axis.
(b) State where $y = [f(x)]^2$ meets the x-axis.
(c) State the point that $(0, -4)$ becomes on $y = |f(x)|$.
(d) State the roots of $y = f(|x|)$ and the coordinates of its minimum point.

22. Let $f(x) = x^2 - 2x - 3$. On separate axes, sketch each graph, stating the coordinates of every intercept and turning point.

(a) $y = |f(x)|$
(b) $y = f(|x|)$
(c) $y = [f(x)]^2$

Modulus equations and inequalities

23. Solve each equation.

(a) $|x-3|=2$
(b) $|2x+1|=7$
(c) $|3-x|=1$
(d) $|x|=5$

24. Solve each equation. One part has no solution.

(a) $|x-2|=|2x+1|$
(b) $|3x|=|x+4|$
(c) $|x-1|=3x-5$
(d) $|2x+1|=x-1$

25. Solve each inequality.

(a) $|x-4|<3$
(b) $|2x+5|\geq 9$
(c) $|3-x|\leq 1$
(d) $|x^2-9|<7$

26. GDC Both sides now contain x.

(a) Sketch $y=|x-2|$ and $y=x$ on the same axes, and hence solve $|x-2|<x$.
(b) Solve $|2x|>x+3$ algebraically.
(c) Use a GDC to solve $|x^3-3x|>1$ for $0\leq x\leq 3$, giving answers to three significant figures.

27. Solve $|x^2-4|<3$.

Concept check

28. A student writes: "$f(x)=x^3+1$, *so* $f(-x)=-x^3+1\neq f(x)$. *The function is not even, so it is odd.*" Identify the error and classify f correctly.

29. A student solves $|x-3|=2x$ and gives the answer "$x=-3$ *or* $x=1$". Identify the error and give the correct solution.

5.4 Reciprocal and Rational Functions

A line, a parabola and a cubic have a value at every input, and each one rises or falls without limit. Divide by a variable and both of those stop being true. There are now inputs where the function has no value, and there are lines, called asymptotes, that the curve approaches. Finding those lines from the formula is what makes a rational curve quick to sketch. The same work also tells you where the function is positive and where it is negative.

The simplest function of this kind is the **reciprocal** $y=\dfrac{1}{x}$, and it shows all of the new behaviour at once.

It is undefined at $x=0$. As x approaches 0, the output grows without bound, towards $+\infty$

on one side and $-\infty$ on the other. For large $|x|$ the opposite happens: the output becomes very small, and the curve approaches zero. So the reciprocal has two asymptotes, the vertical line $x = 0$ and the horizontal line $y = 0$.

It is defined for every input except $x = 0$, so its **domain** is all real numbers with $x \neq 0$. It produces every output except 0, so its **range** is all real numbers with $y \neq 0$. It is also self-inverse, so its graph is symmetric in the line $y = x$. See §5.3.

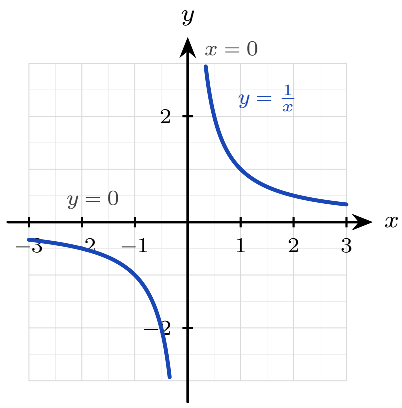


Figure 5.12 The reciprocal $y = \frac{1}{x}$. Its asymptotes are the two axes: the curve approaches $x = 0$ and $y = 0$ without meeting either. The domain is every real number except 0, and so is the range.

DEF An **asymptote** is a line that the graph of a function approaches arbitrarily closely (as close as you like). A **vertical asymptote** occurs at a value of x where the function is undefined and grows without bound. A **horizontal asymptote** is a value the function approaches as $x \to \pm\infty$.

5.4.1 Linear over Linear

Every function of the form $\dfrac{ax+b}{cx+d}$ is a reciprocal curve that has been shifted and stretched, so all of them have the same two-branch shape. Dividing the numerator by the denominator shows that directly: it rewrites the function as a constant plus a multiple of $\dfrac{1}{x-p}$, and the two asymptotes then cross at the centre of that reciprocal curve. Two conditions are needed for that to be true. If $c = 0$ the function is linear and has no asymptote at all. If $ad - bc = 0$ the numerator is a multiple of the denominator and the function is constant.

KEY For $f(x) = \dfrac{ax+b}{cx+d}$, with $c \neq 0$ and $ad - bc \neq 0$:

$$\text{vertical asymptote at } x = -\frac{d}{c}, \qquad \text{horizontal asymptote at } y = \frac{a}{c}.$$

The vertical one is where the denominator is zero. The horizontal one is the ratio of the leading coefficients.

Worked through. To sketch $f(x) = \dfrac{2x+1}{x-1}$, find the vertical asymptote first. The denominator is zero at $x = 1$, so the line $x = 1$ is the vertical asymptote. Next the horizontal asymptote. The ratio of the leading coefficients gives $y = \dfrac{2}{1} = 2$. Rewriting the function confirms the shape. Divide the numerator by the denominator:

$$\frac{2x+1}{x-1} = 2 + \frac{3}{x-1},$$

a reciprocal curve centred on the crossing of the two asymptotes, $(1, 2)$.

Now find the intercepts. Substitute $x = 0$ for the y-intercept: $f(0) = \dfrac{1}{-1} = -1$. Set the numerator to zero for the x-intercept: $2x + 1 = 0$, so $x = -\frac{1}{2}$. Both are marked on the figure below. Two asymptotes and two intercepts fix the whole sketch.

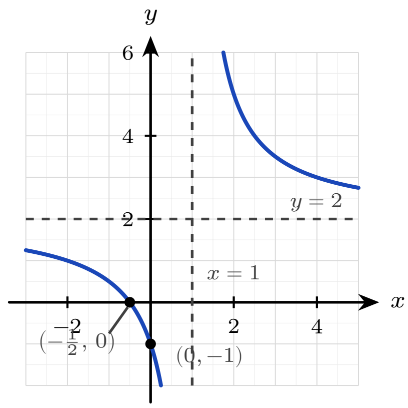


Figure 5.13 The curve $y = \frac{2x+1}{x-1}$, with its two asymptotes dashed and its two intercepts marked. These four labelled features are what a sketch must show.

TIP Marks for a sketch usually depend on labelled asymptotes and intercepts. Write the equation of each asymptote beside its dashed line, and write the coordinates beside each intercept.

The divided form answers a second question, and it answers it for any quotient of two linear expressions. A sketch shows which heights the curve reaches, but a question asking for the range wants a reason rather than a picture, and writing the function as a constant plus a fraction supplies one. Written that way, the function can never equal the constant, because the fraction is never zero. Every other value is reached, which you show by setting $f(x) = k$ and solving for x. So the range of $\dfrac{ax+b}{cx+d}$, with $c \neq 0$ and $ad \neq bc$, is every real number except the horizontal asymptote, $y = \dfrac{a}{c}$. The function below is the one whose inverse was found in §5.1.4.

EXAMPLE 5.13

State the range of $f(x) = \dfrac{3x-1}{x-1}$, $x \neq 1$, justifying your answer.

Divide the numerator by the denominator, the same first move as in the sketch above:

$$f(x) = 3 + \frac{2}{x-1}.$$

The second term is a fraction with numerator 2, so it is never 0, whatever x is. Adding something that is never 0 to 3 never gives 3, so $f(x) \neq 3$ for every x in the domain. Every other value is produced. Set $f(x) = k$ and solve back for x:

$$\frac{2}{x-1} = k - 3 \implies x - 1 = \frac{2}{k-3} \implies x = 1 + \frac{2}{k-3},$$

which is a real input, and never equal to 1, whenever $k \neq 3$. So the range is $y \in \mathbb{R}$, $y \neq 3$. The one excluded value is the horizontal asymptote. A linear-over-linear curve never takes that value, so this curve never reaches $y = 3$.

5.4.2 Reading the Asymptotes from the Degrees

The two rules above are one case of a general pattern. Which asymptote a rational function has as $x \to \pm\infty$ depends only on how the degree of the numerator compares with the degree of the denominator.

Degrees	Asymptote as $x \to \pm\infty$	Example
numerator $<$ denominator	horizontal, $y = 0$	$\frac{x+1}{x^2-4}$
numerator $=$ denominator	horizontal, at the ratio of leading coefficients	$\frac{2x+1}{x-1}$
numerator $=$ denominator $+1$	oblique, a slanted line	$\frac{x^2-2x+3}{x-1}$

In every case the vertical asymptotes sit at the zeros of the denominator, once any common factor has been cancelled.

A horizontal or oblique asymptote describes the curve only for large $|x|$. Nearer the origin the curve may cross it. A curve cannot cross a vertical asymptote, because the function is undefined there.

The syllabus names one more shape: a linear numerator over a quadratic denominator, $\frac{ax+b}{cx^2+dx+e}$. The denominator has the higher degree, so the horizontal asymptote is $y = 0$, and each real root of the denominator gives its own vertical asymptote. This form can therefore have two.

With two asymptotes and an intercept, the branches can still go several ways. The sign of f on each interval shows which way its branch goes. A rational function changes sign only where it is zero or where it is undefined, so those values cut the x-axis into intervals, and the sign is the same right across each one. Testing one point per interval is therefore enough.

The sign beside a vertical asymptote gives the direction of that branch. Where f is positive the curve climbs to $+\infty$ as it nears the asymptote; where f is negative it falls to $-\infty$.
§5.4.5 sets the same work out as a table.

Worked through. To sketch the key features of $f(x) = \dfrac{x+1}{x^2-4}$, factorise the denominator to find the vertical asymptotes: $x^2 - 4 = (x-2)(x+2)$, so there are *two*, at $x = 2$ and $x = -2$.

The numerator has lower degree than the denominator, so the horizontal asymptote is $y = 0$.

For the intercepts, set the numerator to zero to get $x = -1$, and substitute $x = 0$ to get $f(0) = -\frac{1}{4}$.

Now the signs. The three values -2, -1 and 2 cut the axis into four intervals. Take the sign of each factor of $\dfrac{x+1}{(x-2)(x+2)}$, one interval at a time:

$$x < -2: \ \tfrac{-}{(-)(-)} < 0, \qquad -2 < x < -1: \ \tfrac{-}{(-)(+)} > 0,$$

$$-1 < x < 2: \ \tfrac{+}{(-)(+)} < 0, \qquad x > 2: \ \tfrac{+}{(+)(+)} > 0.$$

Read the branches from those signs. Where f is positive next to a vertical asymptote the curve rises to $+\infty$; where it is negative it falls to $-\infty$. The middle branch is negative to the right of $x = -1$ and positive to the left of it, and it passes through both intercepts. In particular it crosses its horizontal asymptote $y = 0$ at $x = -1$: a horizontal asymptote only describes the behaviour for large $|x|$.

A zero of the denominator does not always produce an asymptote. That is what the phrase "once any common factor has been cancelled" is guarding against. If the same factor divides the numerator too, it cancels, and all that is left at that value of x is one missing point: a **hole** (a removable discontinuity). The function still has no value there, so the domain still excludes it, but the curve on either side lines up.

EXAMPLE 5.14

Describe the behaviour of $f(x) = \dfrac{x^2-9}{x-3}$ at $x = 3$, and state its domain. Compare it with $g(x) = \dfrac{x^2-9}{x+4}$ at $x = -4$.
Factorise the numerator first, because that is what decides the answer:

$$x^2 - 9 = (x+3)(x-3).$$

For f the factor $x - 3$ appears above and below the line, so it cancels:

$$f(x) = \frac{(x+3)(x-3)}{x-3} = x + 3, \qquad x \neq 3.$$

The graph of f is therefore the straight line $y = x + 3$ with a single point removed. That point is at $x = 3$, where the line has height $3 + 3 = 6$, so the hole is at $(3, 6)$. There is no vertical

asymptote at all, and the domain is $x \in \mathbb{R}$, $x \neq 3$.
For g the denominator is zero at $x = -4$. Substitute that into the numerator: $(-4+3)(-4-3) = 7$, which is not 0, so nothing cancels. As x approaches -4 the numerator approaches 7 while the denominator approaches 0, so $|g(x)|$ grows without bound and $x = -4$ is a vertical asymptote.
That substitution decides which of the two you have: put the denominator's zero into the numerator, and if the result is not 0 then no common factor exists and the asymptote stands.

5.4.3 Oblique Asymptotes HL

When the numerator has a higher degree than the denominator, there is no horizontal asymptote. The curve approaches a slanted line instead, and finding that line is a division problem.

The useful case is when the numerator's degree is exactly one more than the denominator's. **Polynomial division** then splits the function into a linear part plus a remainder, and that remainder approaches zero for large $|x|$. The linear part is the **oblique** (slant) asymptote. Polynomial division is treated in full in Ch. 6; all that is needed here is the quotient-plus-remainder form.

The simplest case is $y = x + \dfrac{1}{x}$. It is undefined at $x = 0$, so $x = 0$ is a vertical asymptote. For large $|x|$ the term $\dfrac{1}{x}$ approaches zero, so the curve approaches the oblique asymptote $y = x$.

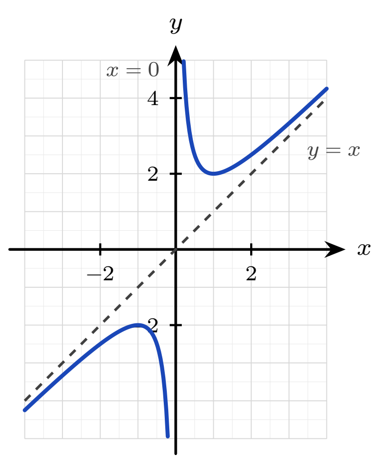


Figure 5.14 The two branches of $y = x + \frac{1}{x}$, each approaching the vertical asymptote $x = 0$ on one side and the oblique asymptote $y = x$ on the other.

There the split into a line plus a remainder was already written out. In general you produce it yourself by **polynomial division**.

Worked through. To find all asymptotes of $f(x) = \dfrac{x^2 - 2x + 3}{x - 1}$, start from the denominator. It is zero at $x = 1$, so there is a vertical asymptote there. The numerator's degree exceeds the denominator's by one, so there is an oblique asymptote, found by dividing:

$$\frac{x^2 - 2x + 3}{x - 1} = x - 1 + \frac{2}{x - 1},$$

since $(x-1)(x-1) + 2 = x^2 - 2x + 3$. As $|x|$ grows the remainder term $\frac{2}{x-1}$ approaches zero, so the curve approaches the line

$$y = x - 1,$$

which is the oblique asymptote.

TIP Show the division, not only the answer. The quotient *is* the asymptote, and the division itself earns method marks.

5.4.4 The Reciprocal of Any Function HL

Every reciprocal so far has been the reciprocal of something simple: of x, of a linear expression, of one polynomial divided by another. The same reading works for the reciprocal of any function at all. Given a function f, its **reciprocal** is

$$y = \frac{1}{f(x)},$$

defined at every x where f is defined and $f(x) \neq 0$.

You do not need to calculate points to sketch it. Every feature of the reciprocal can be read from the graph of f, one feature at a time.

Where f does this	The reciprocal does this	Reason
$f(x) = 0$	vertical asymptote	you cannot divide by zero
$\lvert f(x)\rvert$ is large	the curve approaches the x-axis	a large number has a small reciprocal
$f(x) = 1$ or -1	the two graphs meet	1 and -1 are their own reciprocals
$f(x) > 0$	$\frac{1}{f(x)} > 0$	the sign is unchanged
a local maximum	a local minimum	and a minimum becomes a maximum

The last row needs a second look. Where f is large the reciprocal is small, so a high point of f becomes a low point of $\frac{1}{f}$. This holds wherever f is not zero, whatever the sign of f.

CAUTION $\dfrac{1}{f(x)}$ is *not* $f^{-1}(x)$. The reciprocal divides 1 by the output; the inverse undoes the function. For $f(x) = x + 3$ the reciprocal is $\dfrac{1}{x+3}$ and the inverse is $x - 3$, which are not the same. The notation f^{-1} never means "one over f".

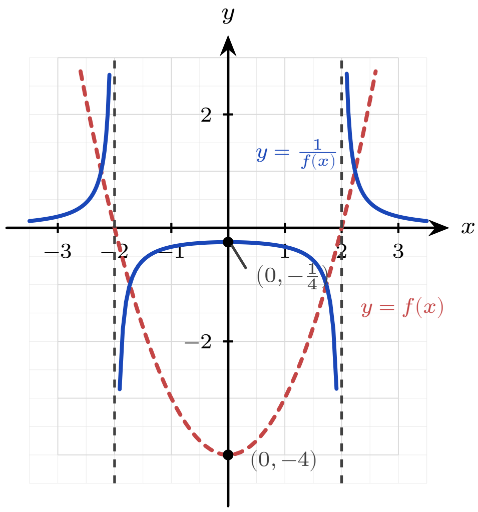


Figure 5.15 $y = \dfrac{1}{x^2 - 4}$ against $y = x^2 - 4$. Zeros of the original become vertical asymptotes at $x = \pm 2$, the minimum $(0, -4)$ becomes the maximum $(0, -\frac{1}{4})$ of the middle branch, and large values of f become values close to zero.

EXAMPLE 5.15

The graph of $y = f(x)$ touches the x-axis at $x = 5$ and has no other root, it has a local minimum at $(2, 4)$, and $f(x) \to +\infty$ as $x \to \pm\infty$. Describe the corresponding features of $y = \dfrac{1}{f(x)}$.

Take the features one at a time.

The root at $x = 5$ makes the denominator zero, so $y = \dfrac{1}{f(x)}$ has a **vertical asymptote at** $x = 5$. Since the curve only touches the axis there, f is positive on both sides of $x = 5$, so the reciprocal runs up to $+\infty$ on both sides of that asymptote.

The minimum at $(2, 4)$ has $f(2) = 4$, so the reciprocal passes through $(2, \frac{1}{4})$. A minimum of f becomes a **maximum** of the reciprocal, so this is a local maximum at $(2, \frac{1}{4})$.

Since $f(x) \to \infty$ at both ends, $\dfrac{1}{f(x)} \to 0$, so $y = 0$ is a **horizontal asymptote**.

No algebra was needed. Every answer came from one feature of f.

TIP Sketching a reciprocal in an examination: mark the roots of f as vertical asymptotes first, then the points where $f = 1$ and $f = -1$, then the turning points with their y-coordinates inverted. Draw the curve through those, one branch at a time.

5.4.5 Rational Inequalities HL

An inequality asks the same question as a sketch, in a shorter form: on which intervals is the expression positive, and on which is it negative?

One move gets it into a shape you can answer. Collect every term on one side, so that the inequality compares a single fraction with zero.

The rest is a **sign table**. A rational expression changes sign only where it is zero or where it is undefined, so those **critical values** cut the x-axis into intervals on which the sign never changes. Give the table one row for each factor and one column for each interval. Fill in the sign of every factor, then multiply down each column to get the sign of the whole expression. The last row is the answer, read off interval by interval.

KEY HL To solve a rational inequality:

1. Collect every term on one side, combine into a single fraction and factorise.
2. Mark the **critical values**, where the numerator or the denominator is zero.
3. Build a sign table on the intervals between them, and read off the intervals with the sign you need.
4. Include a zero of the numerator when the inequality is $\geq$ or $\leq$; never include a zero of the denominator.

CAUTION Do not multiply both sides by an expression whose sign you do not know. Multiplying an inequality by a negative number reverses it, so multiplying by $x-2$ would split the question into two cases, one for each sign of $x-2$. Move everything to one side instead.

The same table works when there is no fraction at all, so the quadratic inequalities of Ch. 4 can be solved this way too.

Worked through. To solve $\dfrac{x+3}{x-2} \geq 2$, collect everything on one side and combine into a single fraction:

$$\frac{x+3}{x-2} - 2 \geq 0 \implies \frac{(x+3)-2(x-2)}{x-2} \geq 0 \implies \frac{7-x}{x-2} \geq 0.$$

Now the critical values. The numerator is zero when $7-x=0$, that is $x=7$. The denominator is zero when $x=2$. These two values cut the axis into three intervals.

Interval	$x<2$	$2<x<7$	$x>7$
$7-x$	$+$	$+$	$-$
$x-2$	$-$	$+$	$+$
$\frac{7-x}{x-2}$	$-$	$+$	$-$

Only the middle interval gives the sign required. Now decide the two endpoints. Exclude $x=2$, because the fraction is undefined there. Include $x=7$, because the inequality allows equality and the fraction is zero there. So

$$2 < x \leq 7.$$

A graph agrees: $y = \dfrac{x+3}{x-2}$ lies above the line $y=2$ only between the asymptote and the crossing point. The boundary of the solution is a vertical asymptote at one end and a crossing at the other, and a sign table finds both.

TIP A GDC can solve an equation or inequality of this kind numerically once you move everything to one side. Rewrite $f(x) = g(x)$ as $f(x) - g(x) = 0$, graph the single function $y = f(x) - g(x)$, and read its zeros; for an inequality, read the intervals where that curve is above or below the axis. Reading zeros is another method, an alternative to locating two intersection points, and either method works when an equation has no exact solution, such as $e^x = 3 - x$.

YOUR TURN 5.4 Reciprocal and Rational Functions

Linear over linear

30. State the vertical and horizontal asymptotes of each function.

(a) $f(x) = \dfrac{1}{x-3}$ (b) $f(x) = \dfrac{3}{x+2}$

(c) $f(x) = \dfrac{2x}{x+1}$ (d) $f(x) = \dfrac{x+1}{x-2}$

(e) $f(x) = \dfrac{5}{2x-1}$ (f) $f(x) = \dfrac{4x+1}{2x-3}$

31. For each function, state the domain, the range, both asymptotes and both intercepts. Where an intercept does not exist, say why.

(a) $f(x) = \dfrac{1}{x-5}$

(b) $f(x) = \dfrac{1}{x} + 4$

(c) $f(x) = \dfrac{x-1}{x+2}$

(d) $f(x) = \dfrac{2x+3}{x-1}$

32. Consider $f(x) = \dfrac{3x-6}{x+1}$.

(a) Write $f(x)$ in the form $A + \dfrac{B}{x+1}$.

(b) Hence state the point at which the two asymptotes cross.

(c) Sketch the curve, showing both asymptotes and both intercepts.

Asymptotes and holes

33. For each function, state every vertical asymptote, and state the kind and the equation of the asymptote as $x \to \pm\infty$.

(a) $f(x) = \dfrac{1}{(x-1)(x+3)}$

(b) $f(x) = \dfrac{x}{x^2-9}$

(c) $f(x) = \dfrac{x^2+1}{x^2-4}$

(d) $f(x) = \dfrac{x^2+1}{x-2}$

(e) $f(x) = \dfrac{x^2-9}{x+2}$

34. For each function, describe the graph at every value of x where the denominator is zero, and state the domain.

(a) $f(x) = \dfrac{x^2-4}{x-2}$

(b) $f(x) = \dfrac{x^2+x-6}{x+3}$

(c) $f(x) = \dfrac{x^2-1}{x^2+x}$

Oblique asymptotes

35. Write each function as a polynomial plus a remainder, and state the oblique asymptote.

(a) $f(x) = \dfrac{x^2+3}{x}$

(b) $f(x) = \dfrac{x^2}{x-1}$

(c) $f(x) = \dfrac{2x^2-x+3}{x-1}$

(d) $f(x) = \dfrac{x^2+4x+1}{x+1}$

36. Consider $f(x) = x + \dfrac{2}{x}$.

(a) State the vertical asymptote.

(b) State the oblique asymptote.

(c) Explain why the curve never crosses the oblique asymptote.

(d) Sketch the two branches.

37. Consider $f(x) = x + \dfrac{1}{x}$.

(a) State the vertical asymptote.

(b) State the oblique asymptote.

(c) State whether each branch lies above or below the oblique asymptote, and give a reason.

The reciprocal of a function

38. For each function f, sketch $y = \dfrac{1}{f(x)}$, showing every asymptote and the coordinates of any turning point.

(a) $f(x) = x - 4$
(b) $f(x) = x^2 - 1$
(c) $f(x) = x^2 + 1$
(d) $f(x) = (x-1)(x-3)$

39. The graph of $y = f(x)$ crosses the x-axis at $x = -2$ and at $x = 2$ and has no other root. It has a local maximum at $(0, 4)$, and $f(x) \to -\infty$ as $x \to \pm\infty$.

(a) State the equations of the vertical asymptotes of $y = \dfrac{1}{f(x)}$.

(b) Find the coordinates of the turning point of $y = \dfrac{1}{f(x)}$ at $x = 0$, and state its type.

(c) Describe the behaviour of $y = \dfrac{1}{f(x)}$ as $x \to \pm\infty$.

(d) Given that $f(x) = 4 - x^2$, find the x-coordinates of the points where $y = f(x)$ and $y = \dfrac{1}{f(x)}$ meet.

40. Let $f(x) = x - 2$.

(a) On one set of axes, sketch $y = f(x)$ and $y = \dfrac{1}{f(x)}$.

(b) Find the coordinates of the two points where the graphs meet. *(Hint: they meet where $f(x) = \pm 1$)*

(c) State the interval on which $\dfrac{1}{f(x)} > f(x)$ for $x > 2$.

Rational inequalities

41. Solve each inequality. HL

(a) $x^2 \leq x + 6$
(b) $x^2 > 2x + 3$
(c) $\dfrac{x-4}{x+1} \geq 0$
(d) $\dfrac{3}{x-2} > 1$
(e) $\dfrac{x+2}{x-1} \geq 3$
(f) $\dfrac{2x-1}{x+2} \leq 1$

42. Solve $x^2 \leq 3x + 10$ using a sign table.

Concept check

43. A student solves $\dfrac{x+1}{x-3} \geq 2$ by multiplying both sides by $x - 3$: "$x + 1 \geq 2x - 6$, *so* $x \leq 7$." Identify the error and give the correct solution.

44. A student says: "*a graph never crosses its horizontal asymptote.*" Give a counter-example, and explain why the claim is false.

5.5 Exponential and Logarithmic Functions

Pour a cup of coffee at $90\,^\circ$C and set it down in a room at $20\,^\circ$C. It cools quickly at first, then more slowly, then so slowly that you can barely measure the change. Look at the numbers rather than the coffee. In each fixed interval of time the difference between the cup and the room falls by the same *fraction*, not by the same amount. Isaac Newton measured that and found the rule. No polynomial behaves this way.

An **exponential function** $y = a^x$ does, because it places the variable in the exponent rather than in the base. That one change has a large consequence: an exponential grows at a rate proportional to its own size, so a quantity twice as large grows twice as fast. The same function therefore describes cooling coffee, growing populations and decaying atoms. Ch. 1 sets up the algebra of exponents and logarithms. This section is about their graphs.

For $a > 1$ the graph rises, and rises ever more steeply: this is **growth**. For $0 < a < 1$ it falls towards zero: this is **decay**. Either way the graph passes through $(0, 1)$, stays positive everywhere, and approaches the horizontal asymptote $y = 0$ on one side. The domain is all real numbers and the range is $y > 0$.

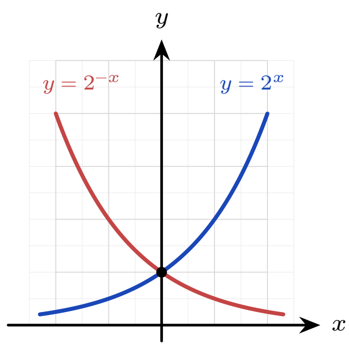


Figure 5.16 Growth and decay are reflections of each other in the y-axis. Both pass through $(0, 1)$ and approach $y = 0$, but on opposite sides.

One base is used most often in calculus: $e \approx 2.718$, the base whose curve has steepness equal to its height at every point. That property makes the calculus of e^x the simplest of any exponential. It is also why every other exponential is usually rewritten in terms of e. See Ch. 13.

> KEY · IB Any exponential can be written with base e, and exponential and logarithm undo each other:
>
> $$a^x = e^{x \ln a}, \qquad \log_a(a^x) = x, \qquad a^{\log_a x} = x \quad (a, x > 0,\ a \neq 1).$$

Worked through. For $f(x) = 5(0.4)^x$, look at the base first. Here $a = 0.4$, and $0 < 0.4 < 1$, so the function **decays**. The factor 5 stretches the graph vertically and changes nothing else, and the outputs of $(0.4)^x$ are always positive. So the domain is all real numbers, the range is $y > 0$, and the horizontal asymptote is $y = 0$.

5.5.1 Transforming an Exponential Graph

An exponential often appears in a model as

$$y = k\,a^x + c,$$

and each constant has a separate effect. The factor k stretches the graph vertically. The constant c moves it up or down.

The vertical shift is the one that changes the asymptote. The graph of a^x approaches $y = 0$, so adding c makes it approach $y = c$ instead. **The horizontal asymptote moves with the graph.**

Worked through. To sketch $y = 3(2)^x - 5$, stating the asymptote and both intercepts, start with the asymptote. The -5 shifts the whole graph down by 5, so the horizontal asymptote is $y = -5$, not $y = 0$.

For the y-intercept, substitute $x = 0$:

$$y = 3(2)^0 - 5 = 3 - 5 = -2.$$

For the x-intercept, set $y = 0$ and solve:

$$3(2)^x = 5 \implies 2^x = \tfrac{5}{3} \implies x = \log_2 \tfrac{5}{3} \approx 0.737.$$

The curve therefore rises from just above $y = -5$ on the left. It crosses the axes at $(0, -2)$ and at $(0.737, 0)$, then rises steeply.

CAUTION A shifted exponential does *not* have the asymptote $y = 0$. In $y = k\,a^x + c$ the asymptote is $y = c$. Reading it as $y = 0$ without checking is a common error.

5.5.2 Exponential Models

Many real quantities are modelled with base e and time as the variable:

$$A(t) = A_0\,e^{kt},$$

where A_0 is the amount at $t = 0$. The sign of k decides the behaviour, $k > 0$ for growth and $k < 0$ for decay, and its size decides the speed. Ch. 1 solves these models algebraically; what is added here is reading them off a graph.

Two data points fix the whole model. If one point is at $t = 0$, it gives A_0 directly. For the second, substitute its t and A into $A = A_0 e^{kt}$, divide by A_0 and take logarithms:

$$k = \frac{1}{t} \ln\left(\frac{A}{A_0}\right).$$

CAUTION A model fitted to two points assumes the fractional rate of change never varies. Where the real rate is itself changing, the prediction becomes less reliable, so report such a figure as what would happen if nothing changed rather than as a forecast.

Worked through. To model Japan's population, about 128.1 million in 2010 and about 126.1 million in 2020, as $P(t) = P_0 e^{kt}$ with t in years after 2010, and to use the model to estimate the population in 2030, take $P_0 = 128.1$, the value at $t = 0$, then use the 2020 figure to find k. Substitute $t = 10$ and $P = 126.1$:

$$126.1 = 128.1\, e^{10k} \implies k = \frac{1}{10} \ln\left(\frac{126.1}{128.1}\right) \approx -0.00157.$$

The sign is negative, which confirms decay: the population falls by about 0.16% each year. For 2030, substitute $t = 20$:

$$P(20) = 128.1\, e^{-0.00157 \times 20} \approx 124.1 \text{ million}.$$

Japan's rate of decline is itself increasing, so official projections for 2030 are lower than this figure.

5.5.3 The Logarithm as an Inverse

Since $y = a^x$ is one-to-one, it has an inverse, and that inverse is the **logarithm** $y = \log_a x$. It answers the reversed question: not "what is a to the x?" but "to what power must I raise a to get x?" Everything about its graph follows from being the mirror image of the exponential in the line $y = x$.

Reflecting swaps the axes' roles. The exponential's horizontal asymptote $y = 0$ becomes the logarithm's *vertical* asymptote $x = 0$; its domain of all reals becomes the logarithm's range, and its range of positive numbers becomes the logarithm's domain. So $\log_a x$ has domain $x > 0$ and range all real numbers; for $a > 1$ it passes through $(1, 0)$ and rises without bound, ever more slowly.

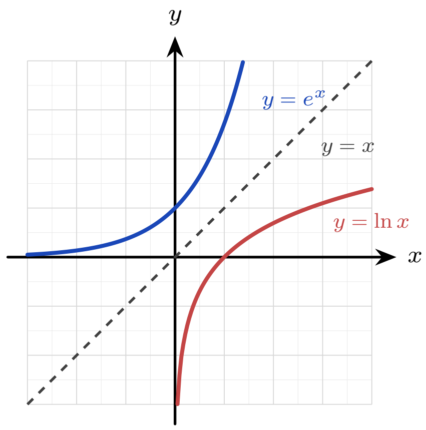


Figure 5.17 The logarithm is the exponential reflected in the line $y = x$. Reflecting exchanges the two axes, which is why one asymptote is horizontal and the other vertical.

CAUTION The logarithm is only defined for positive inputs. Writing $\ln(x-3)$ requires $x > 3$; forgetting this domain restriction is a common error.

FINDING the domain

Because a logarithm accepts only positive inputs, every logarithmic function carries a hidden condition. To find the domain, set whatever sits inside the logarithm greater than zero, and solve.

EXAMPLE 5.16

State the domain of $f(x) = \ln(x-3)$ and of $g(x) = \log(5-2x)$.
For f, the input to the logarithm is $x-3$. Require it to be positive:

$$x - 3 > 0 \implies x > 3.$$

For g, the input is $5-2x$. Require that to be positive:

$$5 - 2x > 0 \implies 5 > 2x \implies x < \tfrac{5}{2}.$$

Note the direction. Dividing by -2 would have reversed the inequality, so it is safer to move the term across as above.

TRANSFORMING a logarithmic graph

The same shifts apply as for any function. A horizontal shift moves the *vertical* asymptote; a vertical shift leaves it where it is.

In $y = \ln(x-h)$ the change is inside the bracket, so it moves the graph h units right. The asymptote travels with it, from $x = 0$ to $x = h$.

The x-intercept moves with it too, and one fact locates it. A logarithm is zero exactly when its input is 1, so $\ln(x-h) = 0$ when $x - h = 1$, that is at $x = h + 1$.

EXAMPLE 5.17

Sketch $y = \ln(x-5)$, stating its asymptote, its domain and its x-intercept.

The input must be positive, so $x - 5 > 0$ and the domain is $x > 5$.

That same condition gives the asymptote. As x approaches 5 from above, $x - 5$ approaches 0 and $\ln(x-5)$ decreases without bound, so $x = 5$ is a vertical asymptote.

For the x-intercept, set $y = 0$. A logarithm is zero when its input is 1:

$$x - 5 = 1 \implies x = 6.$$

So the curve rises from the asymptote at $x = 5$, crosses the axis at $(6, 0)$, and continues upward ever more slowly.

Using the inverse relationship

Because the exponential and the logarithm undo each other, you can use a logarithm to solve for a variable in an exponent, and an exponential to solve for a variable inside a logarithm. That is the step that makes the method of §5.1 work here: at step 3, where you solve for y, apply a logarithm if y sits in an exponent and an exponential if y sits inside a logarithm.

EXAMPLE 5.18

Find the inverse of $f(x) = 2e^{3x} - 5$, and state its domain.

Write $y = f(x)$ and exchange the letters:

$$x = 2e^{3y} - 5 \implies \frac{x+5}{2} = e^{3y}.$$

The variable is now in an exponent, so apply the natural logarithm to both sides:

$$3y = \ln\left(\frac{x+5}{2}\right) \implies f^{-1}(x) = \tfrac{1}{3}\ln\left(\frac{x+5}{2}\right).$$

Now the domain. The domain of f^{-1} is the range of f. Since $e^{3x} > 0$ for every x, we have $2e^{3x} > 0$ and so $f(x) > -5$. The domain of f^{-1} is therefore $x > -5$, which agrees with requiring $\frac{x+5}{2} > 0$ inside the logarithm.

TIP State the domain whenever a question asks for an inverse involving a logarithm. It is a separate mark, and the quickest check is that **the domain of f^{-1} is the range of f**.

YOUR TURN 5.5 Exponential and Logarithmic Functions

Exponential and logarithmic graphs

45. Evaluate each expression.

(a) 2^{-2}
(b) e^0
(c) $\log_2 8$
(d) $\log_5 125$
(e) $\ln 1$
(f) $\ln e$

46. State whether each function grows or decays, and give a reason.

(a) $y = (0.5)^x$
(b) $y = 3^x$
(c) $y = 5e^{-0.2x}$
(d) $y = 2^{-x}$

47. For each graph, state its asymptote and the coordinates of every point where it crosses an axis.

(a) $y = 5 \cdot 2^x$
(b) $y = 2^x + 5$
(c) $y = 4(3)^x - 1$
(d) $y = \ln x$
(e) $y = \ln(x - 2)$
(f) $y = \log(x + 5)$

48. Sketch each graph, stating its asymptote and the exact coordinates of both intercepts.

(a) $y = 2(3)^x - 6$
(b) $y = 8 - 2^{x+1}$
(c) $y = 5e^{-x} - 1$

Base e and exponential equations

49. Write each in the required form.

(a) 2^x in terms of base e
(b) 5^x in terms of base e

50. Solve each equation, giving exact answers.

(a) $2^x = 8$
(b) $\log_3 x = 2$
(c) $5 \cdot 3^x = 45$
(d) $\ln x = 3$
(e) $3(2)^x = 4$
(f) $\log_2(x - 1) = 4$

Domains and inverses

51. State the domain of each function.

(a) $y = \ln(3x + 1)$
(b) $y = \log(10 - 2x)$
(c) $y = \ln(2x - 6)$
(d) $y = \ln(x + 1) + \ln(3 - x)$
(e) $y = \ln(x^2 - 4x)$
(f) $y = \ln(x^2 - 1)$

52. Find the inverse of each function, and state its domain.

(a) $f(x) = e^x + 2$
(b) $f(x) = \ln(x - 1)$
(c) $f(x) = 3e^{2x} + 1$

Exponential models

53. GDC Two exponential models.

(a) A population is modelled by $P(t) = 800e^{0.03t}$, with t in years. State the initial population $P(0)$, and say whether the population grows or decays.
(b) Find the population after 10 years.
(c) Find how long it takes the population to reach 1200.
(d) A radioactive sample decays according to $A(t) = 50e^{-0.2t}$, with t in days and A in grams. Find $A(6)$, correct to three significant figures.
(e) Find the half-life of the sample.

54. GDC A colony of bacteria numbers 500 at $t = 0$ and 1800 after 4 hours. It is modelled by $N(t) = N_0e^{kt}$, with t in hours.

(a) Find the exact value of k, and give it correct to three significant figures.
(b) Estimate the size of the colony after 6 hours.
(c) Find how long it takes the colony to reach $10\,000$.

55. GDC A machine is worth $24\,000$ dollars when new and $15\,000$ dollars three years later. Its value after t years is modelled by $V(t) = V_0e^{kt}$.

(a) Find k, correct to three significant figures, and explain what its sign tells you.
(b) Find the value of the machine after 5 years.
(c) Find the time it takes for the value to halve.

56. GDC After a sauna is switched off, its temperature in $^\circ$C after t minutes is $T(t) = 20 + 70e^{-0.12t}$.

(a) State the temperature at the moment it is switched off.
(b) Find the temperature after 5 minutes.
(c) Find how long it takes to cool to $40\,^\circ$C.
(d) State the temperature the sauna approaches, and explain what it represents.

Equations that need a graph

57. GDC Consider the equation $e^x = 3 - x$.

(a) Explain why it has exactly one real solution.

(b) Find that solution, correct to three significant figures.

(c) State why a graph is the appropriate method for this equation.

Concept check

58. A student states that the graph of $y = 3 \times 2^x - 5$ has the asymptote $y = 0$ and the y-intercept $(0, -5)$. Identify both errors, and give the correct asymptote and y-intercept.

Chapter 5 · Functions & Graphs

Reflections

TOK Slide the graph of $y = x^2$ two units right and you get $y = (x-2)^2$. Is this the same function or a different one? It has the same shape, the same name ("a parabola"), yet a different rule and different values everywhere. When we call two things "the same" in mathematics, we are choosing what to ignore. The transformation makes that choice visible.

IM In 1872 Felix Klein, aged just twenty-three, proposed in his *Erlangen Programme* a radical reorganisation of geometry: a geometry *is* a set of transformations together with the properties they leave unchanged. Euclidean geometry studies what is unchanged by translations, rotations and reflections; other geometries allow other moves. The transformations of this chapter are the same idea in miniature, and the question "what stays the same when I move this graph?" is a central one in mathematics.

RW Exponential decay is used to measure time in physics and archaeology. Every radioactive isotope falls by half over a fixed *half-life*, from carbon-14's $5{,}730$ years, used to date ancient wood and bone, to isotopes that decay in seconds. The same exponential describes unrestricted population growth and compound interest. When growth is limited by a maximum, it becomes the logistic curve of a spreading rumour or an epidemic. Rational functions model saturation: reaction rates, drug concentrations, and average costs that flatten toward a limit.

EXT Some functions undo themselves: $f(f(x)) = x$ for every x in the domain, so $f^{-1} = f$, and the graph is its own reflection in $y = x$. A function like this is called an **involution**. Examples are $f(x) = \dfrac{1}{x}$, $f(x) = 5 - x$ and every $f(x) = \dfrac{ax+b}{cx-a}$ with $c \neq 0$ and $a^2 + bc \neq 0$. The quarter circle $f(x) = \sqrt{1-x^2}$, $0 \leq x \leq 1$, is another. Can you explain from its graph why it must be its own inverse, and find an involution of your own that is neither a line nor a hyperbola?

End-of-Chapter Problems
Chapter 5: Functions & Graphs

1. Let $f(x) = 2x - 3$ and $g(x) = x^2$.

(a) Find $(f \circ g)(x)$. [2]
(b) Find $(g \circ f)(x)$. [2]
(c) Solve $(f \circ g)(x) = 5$. [2]

2. The graph of $y = f(x)$ has a maximum point at $(1, 4)$. State the coordinates of the maximum of:

(a) $y = f(x) + 2$; [1]
(b) $y = f(x - 3)$; [1]
(c) $y = 2f(x)$; [1]
(d) $y = -f(x)$ (state the resulting minimum). [1]

3. Determine whether each function is odd, even, or neither.

(a) $f(x) = x^3 - 2x$ [1]
(b) $f(x) = |x|$ [1]
(c) $f(x) = x^2 + x^3$ [1]
(d) $f(x) = \dfrac{x}{x^2 + 1}$ [1]
(e) $f(x) = \dfrac{1}{x^2 + 1}$ [1]
(f) $f(x) = x + 1$ [1]

4. Solve each inequality.

(a) $x^2 + 2 \leq 3x$ [3]
(b) $|2x - 1| < 5$ [2]
(c) $\dfrac{x - 5}{x + 1} \leq 2$ [4]

5. For each quadratic, write $f(x)$ in completed-square form and describe the transformations that map $y = x^2$ onto $y = f(x)$.

(a) $f(x) = x^2 - 4x + 5$; state also the range of f. [3]
(b) $f(x) = x^2 - 6x$; state also the vertex. [2]

6. The graph of $y = f(x)$ passes through the points $A(0, 1)$, $B(2, 0)$ and $C(3, 4)$.

(a) Find the images of A, B and C on the graph of $y = f(x + 1) - 2$. [2]
(b) Describe the transformations that map $y = f(x)$ onto $y = f(2x - 2)$, and find the images of A, B and C on the graph of $y = f(2x - 2)$. [3]

(c) The graph of $y = f(x)$ is reflected in the x-axis and then translated 3 units up. Write down the equation of the new graph in terms of f, and find the image of C. [2]

7. Solve each equation.

(a) $|x-2| = |x+4|$ [3]
(b) $|x| = 2-x$, and state the coordinates of the point where $y = |x|$ meets $y = 2-x$. [2]
(c) $|x-1| = 3x+5$, showing clearly why one candidate is rejected. [3]

8. Reasoning from the definitions of odd and even.

(a) The function f is even with $f(3) = 7$, and g is odd with $g(2) = -5$. Write down the values of $f(-3)$ and $g(-2)$. [2]
(b) Prove that if f and g are both odd functions, then $f \circ g$ is also odd. [3]

9. Let $f(x) = (x+1)^2 - 4$ for $x \geq -1$.

(a) Explain why f has an inverse on this domain. [1]
(b) Find $f^{-1}(x)$, stating its domain. [4]

10. Let $f(x) = \dfrac{x}{1+x}$, $x \neq -1$.

(a) Find $(f \circ f)(x)$, simplifying your answer fully. [3]
(b) Hence find $(f \circ f \circ f)(x)$. [2]
(c) Find $f^{-1}(x)$. [2]
(d) State the domain of $f \circ f$. [2]

11. Let $g(x) = 3 + \sqrt{x-2}$, $x \geq 2$.

(a) State the range of g. [1]
(b) Find $g^{-1}(x)$, and state its domain. [3]
(c) Sketch the graphs of $y = g(x)$ and $y = g^{-1}(x)$ on the same axes, together with the line $y = x$. [2]
(d) Hence solve $g(x) = g^{-1}(x)$, giving your answer in exact form. [3]

12.

(a) Solve $|x-2| = |3x+1|$. [4]
(b) Hence, or otherwise, solve $|x-2| \geq |3x+1|$. [3]

13. Solve each equation.

(a) $e^{2x} - 3e^x + 2 = 0$ [3]
(b) $4^x - 2^{x+1} = 8$ [4]

14. For each function, state every asymptote, and say what happens at the value of x where the denominator is zero.

(a) $f(x) = \dfrac{x^2-1}{x}$; state also the x-intercepts. [3]

(b) $f(x) = \dfrac{x^2+x-2}{x-1}$; simplify the expression first. [3]

15. The function $f(x) = ax + b$ satisfies $f(f(x)) = x$ for all real x. Find all possible values of a and b. [4]

16. Prove that every function $f : \mathbb{R} \to \mathbb{R}$ can be written as the sum of an even function and an odd function. [4]

17. Show that $f(x) = \dfrac{2x+3}{x-2}$ is self-inverse, that is, $(f \circ f)(x) = x$. [3]

18. Let $f(x) = \dfrac{3x+1}{x-1}$, $x \neq 1$.

(a) Show that $f(x) = 3 + \dfrac{4}{x-1}$. [2]

(b) Hence describe a sequence of transformations that maps the graph of $y = \dfrac{1}{x}$ onto the graph of $y = f(x)$. [3]

(c) Write down the equations of the asymptotes of the graph of f. [2]

(d) Sketch the graph of $y = f(x)$, showing both asymptotes and the coordinates of both intercepts. [3]

19. GDC After a single dose, the amount of a medicine in a patient's bloodstream is modelled by $A(t) = 250(0.82)^t$ mg, where t is the time in hours after the dose.

(a) State the size of the dose, and the percentage of the medicine that leaves the bloodstream each hour. [2]

(b) Find the amount of medicine in the bloodstream after 5 hours. [1]

(c) The medicine is effective while at least 40 mg remains in the bloodstream. Find the length of time for which the dose is effective. [2]

(d) A second patient is given a different medicine, and the amount in their bloodstream is modelled by $B(t) = 400e^{-0.25t}$ mg. Find the time at which the two patients have the same amount of medicine in their bloodstream. [3]

Give your answers to three significant figures.

20. GDC Let $f(x) = \ln(x+3)$ and $g(x) = x^2 - 2x - 1$.

(a) State the domain of f and the equation of the vertical asymptote of its graph. [2]

(b) Sketch the graphs of f and g on the same axes for $-3 < x \leq 4$. [2]

(c) Find the x-coordinates of the points where the two graphs intersect. [2]

(d) Hence solve the inequality $\ln(x+3) > x^2 - 2x - 1$. [2]

Give your answers to three significant figures.

21. GDC Oil leaking from a ship spreads out as a circle. The radius of the circle t hours after the leak starts is $r(t) = 2 + 1.5\sqrt{t}$ km, and the area of a circle of radius r is $A(r) = \pi r^2$.

(a) Find an expression for $(A \circ r)(t)$, and state what it represents. [2]
(b) Find the area of the oil spill after 4 hours. [1]
(c) Find the time at which the area of the oil spill reaches 100 km^2. [2]
(d) Find $r^{-1}(t)$, and state what it represents. [2]

Give your answers to three significant figures.

22. GDC A workshop makes x chairs a month. The average cost of making one chair, in dollars, is modelled by

$$C(x) = \frac{0.02x^2 + 15x + 2000}{x}, \qquad x > 0.$$

(a) Write down the equation of the oblique asymptote of the graph of C, and state what the vertical asymptote $x = 0$ means in context. [2]
(b) Find the number of chairs the workshop should make each month to minimise the average cost, and state the minimum average cost. [3]
(c) Find the numbers of chairs for which the average cost is less than \$30. [2]

23. The curve $y = \dfrac{x^2 + ax + b}{x - 1}$ has oblique asymptote $y = x + 3$.

(a) Find a. [3]
(b) Using your value of a, find the value of b for which the zero of the denominator gives a hole at $x = 1$ instead of a vertical asymptote, and describe the resulting graph. [3]

24. Show that $f(x) = \dfrac{ax + b}{cx + d}$ is self-inverse (that is, $f(f(x)) = x$) if and only if $a + d = 0$, where $c \neq 0$ and $ad - bc \neq 0$. [6]

25. Let $f(x) = \log_3(2x - 1)$ and $g(x) = \frac{1}{2}(3^x + 1)$.

(a) State the domain and the range of f. [2]
(b) Show that g is the inverse of f, by showing that $(f \circ g)(x) = x$ for every real x. [3]
(c) State the equation of the line in which the graphs of f and g are reflections of each other, and write down the coordinates of the point where the graph of g crosses the y-axis. [2]

26. Let $f(x) = x^2 - 4x + 3$.

(a) Write $f(x)$ in completed-square form, and state the coordinates of the vertex. [3]
(b) Find the roots of f and the y-intercept. [3]
(c) Sketch $y = f(x)$, showing the vertex and both intercepts. [2]
(d) On separate axes, sketch $y = |f(x)|$ and $y = f(|x|)$, showing the key points of each. [4]
(e) The graph of $y = f(x)$ is translated 2 units to the right and 3 units up to give $y = g(x)$. Find $g(x)$ in the form $ax^2 + bx + c$. [3]
(f) Describe the transformations that map $y = f(x)$ onto $y = f(2x - 1)$. Hence write down the roots of $f(2x - 1) = 0$ and the coordinates of the vertex of $y = f(2x - 1)$. [3]

27. Let $f(x) = \dfrac{3x - 1}{x + 2}$ for $x \neq -2$, and let $g(x) = x + 2$.

(a) State the range of f, justifying your answer. [2]
(b) Find $f^{-1}(x)$, and state its domain. [3]
(c) Find $(f \circ g)(x)$, giving your answer as a single fraction. [3]
(d) State the domain of $f \circ g$. [2]
(e) Solve $(f \circ g)(x) = 2$. [3]

28. HL Let $f(x) = x^2 - 4$.

(a) Sketch $y = f(x)$, showing the intercepts. [2]
(b) Write down the equations of the vertical asymptotes of $y = \dfrac{1}{f(x)}$, and explain how you found them. [3]
(c) Find the coordinates of the maximum point of $y = \dfrac{1}{f(x)}$ on the interval $-2 < x < 2$, and explain why it is a maximum even though the value is negative. [3]
(d) Describe the behaviour of $y = \dfrac{1}{f(x)}$ as $x \to \pm\infty$, and state the horizontal asymptote. [2]
(e) Sketch $y = \dfrac{1}{f(x)}$, showing all asymptotes and the point found in part (c). [4]

29. HL Consider $f(x) = \dfrac{x^2 - 4x + 5}{x - 2}$.

(a) Show that $f(x) = (x - 2) + \dfrac{1}{x - 2}$. [2]
(b) Write down the equations of the vertical and oblique asymptotes. [3]
(c) Find the y-intercept, and show that the curve has no x-intercept. [3]
(d) Show that the curve never meets its oblique asymptote. [2]
(e) Sketch the curve, showing both asymptotes and the y-intercept. [3]

30. Let $f(x) = \ln(2x - 4)$.

(a) State the domain of f. [2]
(b) Find the exact x-intercept of the graph of f. [2]
(c) State the equation of the vertical asymptote, and describe the behaviour of $f(x)$ near it. [3]

(d) Find $f^{-1}(x)$, and state its domain and range. [3]

(e) Describe a sequence of transformations that maps $y = \ln x$ onto $y = \ln(2x - 4)$. [3]

31. Let $f(x) = x^2 - 4x$.

(a) Sketch $y = |f(x)|$ for $-1 \leq x \leq 5$, showing the coordinates of both corners and of the turning point. [3]

(b) Hence write down the values of k for which the equation $|f(x)| = k$ has exactly four solutions. [2]

(c) Solve $|x^2 - 4x| = 3$. [3]

(d) On separate axes, sketch $y = [f(x)]^2$, stating where the curve meets the x-axis and the coordinates of its turning points. [3]

32. GDC The number of people who have heard a rumour t days after it starts is modelled by

$$N(t) = \frac{5000}{1 + 49e^{-0.8t}}.$$

(a) Find the number of people who had heard the rumour at $t = 0$. [2]

(b) Find $N(5)$, giving your answer to the nearest whole number. [2]

(c) Find the time at which 2500 people have heard the rumour, giving an exact answer and a value to three significant figures. [4]

(d) State the value N approaches as t becomes large, and interpret it in context. [3]

(e) Show that $\dfrac{5000 - N}{N} = 49e^{-0.8t}$, and hence explain why plotting $\ln\left(\dfrac{5000 - N}{N}\right)$ against t gives a straight line. State its gradient and its vertical intercept. [5]

Challenge Questions

33. Self-inverse rational functions. Throughout, let

$$f(x) = \frac{ax + b}{cx + d}, \qquad c \neq 0, \quad ad - bc \neq 0.$$

The function is called *self-inverse* when $(f \circ f)(x) = x$ for every x in its domain. This investigation finds exactly which of these functions are self-inverse, and then which return to x after three or four applications.

(a) Show that $f(x) = \dfrac{x + 1}{x - 1}$ is self-inverse. [2]

(b) Show that if $ad - bc = 0$ then f is constant, and explain why the condition $ad - bc \neq 0$ is therefore assumed. [3]

(c) Show that

$$(f \circ f)(x) = \frac{(a^2 + bc)\,x + b(a+d)}{c(a+d)\,x + (bc + d^2)}.$$

[4]

(d) Hence prove that f is self-inverse if and only if $a + d = 0$. *(Hint: two expressions in x are equal for all x only when their coefficients match)* [4]

(e) Let $g(x) = \dfrac{1}{1-x}$. Show that g is not self-inverse, but that $(g \circ g \circ g)(x) = x$ for every $x \neq 0, 1$. [3]

(f) Explain why $(f \circ f \circ f)(x) = x$ for every x in the domain is equivalent to $(f \circ f)(x) = f^{-1}(x)$, where $f^{-1}(x) = \dfrac{dx - b}{-cx + a}$. By comparing this with part (c), prove that, when $a + d \neq 0$, the function f returns to x after three applications if and only if

$$(a+d)^2 = ad - bc.$$

Verify the condition for g. *(Hint: two such fractions, each with a nonzero value of "$ad - bc$", are equal for every x exactly when the four coefficients of one are a constant multiple of the four coefficients of the other)* [5]

(g) Show that $h(x) = \dfrac{x-1}{x+1}$ returns to x after four applications but not after two. Compare $(a+d)^2$ with $ad - bc$ for the self-inverse functions, for g and for h, and suggest a condition for f to return to x after n applications. [3]

34. When does the order of two transformations matter? Section 5.2 showed that a vertical stretch and a vertical translation give different curves in different orders. This investigation finds the exact condition for any two vertical moves, and a way to read that condition from the graph.

A *vertical move* with parameters $a \neq 0$ and k replaces $y = f(x)$ by $y = af(x) + k$: a vertical stretch of factor a followed by a translation of k units up. Throughout, start from $y = f(x)$.

(a) Apply a vertical stretch of factor a and then a translation of k units up, and write down the result. Do the same in the opposite order. Deduce the exact condition on a and k under which the two agree for every function f. [3]

(b) Two vertical moves have parameters (a_1, k_1) and (a_2, k_2). Show that the first followed by the second gives $y = a_1 a_2 f(x) + a_2 k_1 + k_2$, and find the result of the opposite order. Deduce that the two orders give the same curve for every f if and only if

$$k_1(a_2 - 1) = k_2(a_1 - 1).$$

[5]

(c) Check the condition in part (b) for two translations ($a_1 = a_2 = 1$), for two stretches ($k_1 = k_2 = 0$), and for a stretch with a translation, comparing the last with part (a). Then show that the moves with parameters $(2, 1)$ and $(3, 2)$ may be applied in either order, although neither is a pure stretch or a pure translation. [4]

(d) Show that when $a \neq 1$ the move with parameters (a, k) leaves every point on the line

$y = \dfrac{k}{1-a}$ at the same height. Show that, when $a_1 \neq 1$ and $a_2 \neq 1$, the condition in part (b) says exactly that the two moves leave the same horizontal line fixed, and check this for the pair in part (c). [4]

(e) Explain why any vertical move and any horizontal move give the same curve in either order. *(Hint: consider which variable each move acts on)* [2]

(f) A horizontal move replaces x by $bx + c$, where $b \neq 0$. Investigate when two horizontal moves may be applied in either order, and whether a fixed-line rule like the one in part (d) still holds. Suggest one way to extend the investigation further. [3]

35. Counting the solutions of $|f(x)| = k$. Let $f(x) = x^3 - 3x$ and let $k \geq 0$ be a constant. The graph of $y = f(x)$ has a local maximum at $(-1, 2)$ and a local minimum at $(1, -2)$.

(a) Sketch $y = |f(x)|$, showing the x-intercepts in exact form and the coordinates of the local maxima. [3]

(b) Find the number of real solutions of $|f(x)| = k$ for each of $k = 0$, $k = 1$, $k = 2$ and $k = 3$. *(Hint: for $k = 2$, each of $x^3 - 3x - 2$ and $x^3 - 3x + 2$ has a repeated linear factor)* [4]

(c) Explain, using your sketch, why the number of solutions changes at $k = 0$ and $k = 2$ and nowhere else. [3]

(d) State the number of real solutions of $|f(x)| = k$ for every $k \geq 0$, giving your answer as a list of cases. [2]

(e) Now let $g(x) = x^3 - 3x + 1$, whose graph has a local maximum at $(-1, 3)$ and a local minimum at $(1, -1)$. Give the corresponding list of cases for $|g(x)| = k$, and explain why it has more cases than the list in part (d). [4]

(f) State a rule that predicts, from the graph of $y = |g(x)|$, the values of k at which the number of solutions changes. Suggest how you would test your rule on a quartic. [2]

36. GDC **Newton's law of cooling.** A body at temperature T in a room at temperature R cools so that the difference $T - R$ decays exponentially:

$$T(t) = R + (T_0 - R)\, e^{-kt}, \qquad k > 0,$$

where T_0 is the temperature at $t = 0$ and t is measured in minutes.

(a) Show that the temperature difference $T(t) - R$ falls by the same factor over any interval of fixed length, and that this factor does not depend on T_0. [4]

(b) Deduce that the difference halves every $\dfrac{\ln 2}{k}$ minutes. [3]

(c) Three readings T_1, T_2 and T_3 are taken at times t, $t + h$ and $t + 2h$. Use part (a) to show that $(T_2 - R)^2 = (T_1 - R)(T_3 - R)$, and hence that

$$R = \frac{T_1 T_3 - T_2^2}{T_1 + T_3 - 2T_2},$$

provided the denominator is not zero. [5]

(d) A cup of coffee is left in a room whose temperature is not known. It reads $95\,^\circ$C, $65\,^\circ$C and $47\,^\circ$C at $t = 0$, 10 and 20 minutes. Find the room temperature R, and find k in exact form. [4]

(e) Find, to the nearest minute, how long the coffee takes to reach $40\,^\circ$C. [2]

(f) Explain why the model predicts that the coffee never reaches room temperature, and state which feature of the graph of T against t shows this. [2]

(g) Suppose the third reading had been recorded as $48\,^\circ$C instead of $47\,^\circ$C. Find the new estimate of R, and comment on how sensitive the method in part (c) is to a small error in one reading. Suggest how more than three readings could be used to estimate R more reliably. [3]

37. The range of a quadratic over a linear function. Section 5.4 finds the range of $\dfrac{ax+b}{cx+d}$ by writing it as a constant plus a fraction. When the numerator is a quadratic, that method no longer gives the range directly. This investigation uses the discriminant instead. Throughout,

$$f(x) = \frac{x^2 + bx + c}{x + d}, \qquad x \neq -d,$$

where b, c and d are real constants, and $N(x) = x^2 + bx + c$ is the numerator.

(a) Let $f(x) = \dfrac{x^2+3}{x-1}$. Show that $f(x) = y$ leads to $x^2 - yx + (y+3) = 0$. By requiring this equation to have a real root, show that the range of f is $y \leq -2$ or $y \geq 6$, and check that the excluded input $x = 1$ removes no value of y. [5]

(b) For the function in part (a), find the value of x at which $f(x) = -2$ and the value at which $f(x) = 6$, and explain why these points are turning points of the graph. Write $f(x)$ as $x + 1 + \dfrac{4}{x-1}$ and state the equations of the asymptotes. [4]

(c) Find the range of $\dfrac{x^2 - 4x + 3}{x - 2}$. [3]

(d) For $\dfrac{x^2 + x - 2}{x - 1}$, show that the discriminant method gives $(y-3)^2 \geq 0$, which holds for every y. Explain why the range is nevertheless not every real number, and state the range. [4]

(e) Return to the general f. Show that $f(x) = y$ has a real solution when

$$\Delta(y) = y^2 + (4d - 2b)\,y + b^2 - 4c \geq 0,$$

and that $\Delta(y)$, as a quadratic in y, has discriminant $16N(-d)$. Deduce that when $N(-d) < 0$ the range is every real number, and that when $N(-d) > 0$ the range misses an interval. Interpret the condition $N(-d) < 0$ in terms of where the vertical asymptote lies relative to the zeros of N. [5]

(f) Show that when $N(-d) > 0$ the missing interval has width $4\sqrt{N(-d)}$, and check this against part (a). Suggest how the investigation could be extended, for example to $\dfrac{x^2 + bx + c}{x^2 + px + q}$. [3]