FUNCTIONS

CHAPTER 4

Lines & Quadratics

4

SYLLABUS 1.16 2.1–2.4 2.6 2.7 2.12

On 17 January 1995 an earthquake struck near Kobe, in western Japan. The Akashi Kaikyō Bridge was under construction. Its two towers already stood in the strait, but the roadway between them did not yet exist. The earthquake moved the towers apart.

The engineers did not begin the work again. They measured the new gap and changed the design to match it. The bridge opened in 1998 with a main span of 1991 metres, one metre longer than the 1990 originally planned.

The engineers could do this because the shape of the bridge follows from a few measurements through equations. The main cables of a suspension bridge carry a level roadway of even weight, and they hang in the curve called a *parabola*, the same curve as a thrown ball. If one value in the equations changes, every other measurement follows from it.

That is the value of a **function**. A function takes an input and returns exactly one output. A whole structure therefore becomes a single rule, and a rule can be graphed, studied and recalculated. This chapter builds two functions that later chapters use often: lines, which change at a steady rate, and quadratics, which give the cable its curve.

By the end of this chapter you will give the domain, range and inverse of a function, and find the equation of a line from the information you are given. You will solve systems of linear equations, write a quadratic in its factorised, vertex and expanded forms, use the discriminant to count the roots of an equation, find the sum and product of the roots without solving, and solve quadratic inequalities.

4.1 Functions

A function is a rule you can rely on. Give it an input and it returns one definite output, and it returns the same output every time. That is what makes it possible to reason about.

> DEF A **function** f assigns exactly one output $f(x)$ to each input x. The set of inputs it accepts is the **domain**; the set of outputs that actually occur is the **range**.

The phrase "exactly one" is where the restriction lies. A rule that could return two different outputs for the same input is not a function. On a graph this becomes the **vertical line test**: a curve represents a function only if no vertical line crosses it more than once.

The letter f carries no meaning of its own. A function is named for what it measures, and the letter inside the bracket for what it depends on. A height that changes with time is written $h(t)$, a velocity $v(t)$, a cost that depends on how many items you make $C(n)$. Read $C(n)$ as "the cost of making n items", not as C multiplied by n. Examination questions name their functions this way, so read the notation as a sentence rather than as algebra.

That naming points at what functions are usually for. A **mathematical model** is a function chosen to describe something real: the letters stand for measured quantities, and the rule claims that one determines the other. Modelling adds a question to every problem that pure algebra never asks, which is whether the answer means anything.

Two habits follow from that. The domain is set by the situation, not by the algebra alone: $h(t) = 20t - 5t^2$ is defined for every real t, but a stone in flight only exists from the moment it is thrown until it lands, so the model holds on $0 \leq t \leq 4$ and nowhere else. And a value that solves the equation can still be rejected, because a negative length or a negative time is arithmetic the situation does not allow. Every model is an approximation, so ask what it leaves out before trusting it far beyond the data it came from.

4.1.1 Reading the domain

A function need not accept every number. Three operations put limits on what you may substitute. Learn the three and you can read the domain of every function in this chapter.

Where x appears	Condition	Reason
In a denominator	$\neq 0$	you cannot divide by zero
Inside an even root	≥ 0	a negative number has no real square root
Inside a logarithm	> 0	zero and negative numbers have no logarithm

Note the difference between the last two. A square root allows zero, because $\sqrt{0} = 0$. A logarithm does not, because no power of the base gives zero. That single distinction, $\geq$ against $>$, is a common slip in this topic.

Function	Condition	Domain
$\sqrt{x+3}$	$x+3 \geq 0$	$x \geq -3$
$\sqrt{5-x}$	$5-x \geq 0$	$x \leq 5$
$\dfrac{1}{x-2}$	$x-2 \neq 0$	$x \neq 2$
$\ln(x-4)$	$x-4 > 0$	$x > 4$
$\dfrac{1}{\sqrt{x-1}}$	$x-1 > 0$, since it is both under a root and in a denominator	$x > 1$

When a function contains more than one of these, every condition must hold at once. Write them all down, then take the values that satisfy all of them.

Worked through. For $f(x) = \dfrac{\sqrt{x+4}}{x-3}$ there are two restrictions. The square root needs $x+4 \geq 0$, so $x \geq -4$. The denominator needs $x-3 \neq 0$, so $x \neq 3$. Both must hold, so the domain is

$$x \geq -4, \quad x \neq 3.$$

Notice that $x = 3$ satisfies the first condition but fails the second, so it must still be removed.

EXAMPLE 4.1

State the domain of $g(x) = \ln(9 - x^2)$.

A logarithm needs a positive argument, so $9 - x^2 > 0$, that is $x^2 < 9$. This is a quadratic inequality, solved in §4.7: the parabola $x^2 - 9$ lies below the axis between its roots. So

$$-3 < x < 3.$$

Both ends are excluded, because at $x = \pm 3$ the argument is zero and $\ln 0$ does not exist.

CAUTION Solve the condition, do not guess it. For $\sqrt{5-x}$ the domain is $x \leq 5$, not $x \geq 5$. Write the inequality down and rearrange it every time, however obvious it looks.

4.1.2 Reading the range

The range is the set of outputs that actually occur. There is no single rule, but most range arguments use one of four facts, each saying that some expression can never be negative. A square is never negative, so $x^2 \geq 0$; a modulus is never negative, so $|x| \geq 0$; the square root symbol means the non-negative root, so $\sqrt{x} \geq 0$; and a positive base raised to any power stays positive, so $a^x > 0$.

To use them, build the function up from the part you know. Ask what that part can be, then follow the operations one at a time.

Function	Reasoning	Range
$y = x^2 + 1$	$x^2 \geq 0$, then add 1	$y \geq 1$
$y = \lvert x\rvert - 3$	$\lvert x\rvert \geq 0$, then subtract 3	$y \geq -3$
$y = \sqrt{x} + 2$	$\sqrt{x} \geq 0$, then add 2	$y \geq 2$
$y = 3 - \lvert x + 1\rvert$	$\lvert x + 1\rvert \geq 0$, so subtracting it from 3 gives at most 3	$y \leq 3$
$y = -(x-2)^2 + 5$	$(x-2)^2 \geq 0$, and the minus sign turns the inequality round	$y \leq 5$

CAUTION A minus sign in front reverses the direction. Since $(x-2)^2 \geq 0$, it follows that $-(x-2)^2 \leq 0$, so $-(x-2)^2 + 5 \leq 5$. Multiplying an inequality by a negative number always flips the sign.

For a quadratic, complete the square first. The vertex form shows the smallest or largest output immediately.

EXAMPLE 4.2

State the range of $f(x) = x^2 - 12x + 40$.

Complete the square:

$$x^2 - 12x + 40 = (x-6)^2 - 36 + 40 = (x-6)^2 + 4.$$

Now use the first fact. Since $(x-6)^2 \geq 0$, adding 4 gives $f(x) \geq 4$. The range is $y \geq 4$. The smallest value, 4, occurs at $x = 6$, which is the vertex.

EXAMPLE 4.3

State the domain and range of $g(x) = \sqrt{x-2}$.

Domain. The square root needs $x - 2 \geq 0$, so $x \geq 2$.

Range. As x runs from 2 upward, $x - 2$ runs from 0 upward, and so does its square root. So $g(x) \geq 0$.

The domain comes from the inputs the rule permits. The range comes from the outputs those inputs give. Always ask both questions.

TIP On Paper 2, check a range on your GDC. Graph the function and look at the lowest and highest points the curve reaches. The algebra is still expected in the working, but the graph catches a sign error in seconds.

4.1.3 Reading a graph

Most of what you need from a function can be read straight from its graph. Five features matter, and each has its own name.

Feature	What it is	How to find it
x-intercepts (also called zeros, or roots)	the points where the curve meets the x-axis	set $y = 0$ and solve
y-intercept	the point where the curve meets the y-axis	set $x = 0$
Turning points	the local maximum and minimum points, where the curve changes direction	GDC, or the vertex form for a quadratic
Asymptote	a line that the curve gets closer and closer to as x or y becomes very large	look for values the function cannot take
Intersection	a point that lies on both of two graphs	solve the two equations together

Your GDC finds four of these directly. The graph menu has zero, min/max and intersect tools, and evaluating the function at $x = 0$ gives the y-intercept. An asymptote comes from the equation, not from a GDC tool. On Paper 2 these are the intended method. Graph the function, read the feature, and write down what you asked the calculator to find. Many equations have no algebraic solution. For those, the GDC is the method to use, not just a faster one.

Worked through. To find the y-intercept, the zeros and the turning points of $f(x) = x^3 - 2x^2 - 5x + 4$ with a GDC, giving coordinates to three significant figures where they are not exact, begin with the y-intercept, which needs no calculator. Setting $x = 0$ leaves the constant term, so the curve meets the y-axis at $(0, 4)$.

Graph the function over a window wide enough to show every feature; $-3 \leq x \leq 4$ works here. The curve crosses the x-axis three times, so run the zero tool three times, once on each crossing:

$$(-1.86,\ 0), \qquad (0.678,\ 0), \qquad (3.18,\ 0).$$

The maximum and minimum tools return both coordinates at once:

$$\text{local maximum } (-0.786,\ 6.21), \qquad \text{local minimum } (2.12,\ -6.06).$$

The turning points check the zeros. The local maximum lies above the axis and the local minimum below it, so the curve must cross the axis three times, and three is what the zero tool found.

TIP Know the difference between the command terms: **sketch** means show the shape with key features labelled, not to scale; **draw** means an accurate, to-scale plot. A sketch of a parabola needs at least the vertex, the intercepts, and the correct opening direction.

4.1.4 Inverse functions

An **inverse function** reverses the rule. If f sends 3 to 7, then f^{-1} sends 7 back to 3. An inverse exists only when no two inputs give the same output. If two did, the way back would have two answers, and a function may return only one.

DEF The **inverse** f^{-1} of a function f undoes it: $f^{-1}(f(x)) = x$. It exists when f is **one-to-one**, meaning no two inputs share an output (the *horizontal* line test).

The inverse swaps input and output. Two things follow. Its graph is the mirror image of f in the line $y = x$, and its domain and range are those of f exchanged. To find a formula, write $y = f(x)$, swap x and y, then solve for y.

Worked through. To find the inverse of $f(x) = 2x + 3$, write $y = 2x + 3$, then swap x and y:

$$x = 2y + 3.$$

Solving for y,

$$y = \frac{x-3}{2}, \qquad \text{so} \qquad f^{-1}(x) = \frac{x-3}{2}.$$

Check: f doubles and adds three; f^{-1} subtracts three and halves. Each step of f is undone, in reverse order.

When x appears twice, once on top and once underneath, you cannot undo the rule step by step. Swap the letters anyway, clear the fraction, and collect the y terms.

EXAMPLE 4.4

Find f^{-1} for $f(x) = \dfrac{4x+3}{x-2}$, and state its domain.

Write $y = \dfrac{4x+3}{x-2}$ and swap x and y:

$$x = \frac{4y+3}{y-2}.$$

Multiply both sides by $y - 2$ to clear the fraction, then gather every term containing y on one side:

$$x(y-2) = 4y+3 \implies xy - 2x = 4y + 3 \implies xy - 4y = 2x + 3.$$

Both terms on the left carry a factor of y, so factorise and divide:

$$y(x-4) = 2x+3 \implies f^{-1}(x) = \frac{2x+3}{x-4}, \qquad x \neq 4.$$

The domain of f^{-1} is the range of f, and this confirms it: $f(x) = 4$ would need $4x + 3 = 4x - 8$, which no x satisfies, so f never outputs 4.

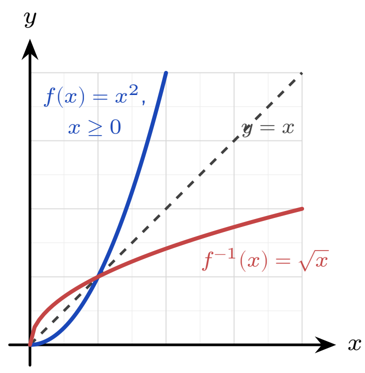


Figure 4.1 A function and its inverse are mirror images in the line $y = x$.

The inverse also gives another way to read an equation. Asking "for which x does $f(x) = 10$?" is the same as asking "what does f^{-1} send 10 to?"

KEY Solving $f(x) = 10$ is equivalent to evaluating $f^{-1}(10)$.

EXAMPLE 4.5

For $f(x) = 2x + 3$, solve $f(x) = 10$ in two ways.
Directly: $2x + 3 = 10$ gives $2x = 7$, so $x = \frac{7}{2}$.
Using the inverse found above, $f^{-1}(x) = \dfrac{x-3}{2}$, so

$$f^{-1}(10) = \frac{10-3}{2} = \frac{7}{2}.$$

The two routes must agree, because they ask the same question. When a question gives you f^{-1} already, the second route is the shorter one.

CAUTION The notation f^{-1} means the inverse, *not* the reciprocal. $f^{-1}(x)$ is not $\dfrac{1}{f(x)}$.

YOUR TURN 4.1 Functions

What a function is

1. For $f(x) = x^2 - 3x$, find each value.

(a) $f(2)$

(b) $f(-1)$

(c) $f(0)$

(d) the values of x with $f(x) = 0$

2. Decide whether each rule defines y as a function of x, and justify your answer.

(a) $y^2 = x$

(b) $y = x^2$

Domain

3. State the domain of each function.

(a) $\sqrt{2x-6}$

(b) $\sqrt{7-x}$

(c) $\dfrac{1}{x+5}$

(d) $\dfrac{1}{x^2-4x}$

(e) $\ln(5-x)$

(f) $\ln(2x+1)$

(g) $\dfrac{1}{\sqrt{x+2}}$

(h) $\sqrt{x^2+1}$

4. State the domain of each function. Each has more than one restriction.

(a) $\dfrac{\sqrt{x+6}}{x-2}$

(b) $\dfrac{\sqrt{x}}{x-9}$

5. A ball is thrown upwards from ground level. Its height after t seconds is $h(t) = 15t - 5t^2$ metres, until it lands.

(a) Find the time at which the ball lands.

(b) State the domain of h in this model.

(c) The value $h(4)$ can be calculated. Explain why it has no meaning in this model.

Range

6. State the range of each function on the domain given.

(a) $x^2 - 4, \quad x \in \mathbb{R}$

(b) $2|x| + 1, \quad x \in \mathbb{R}$

(c) $5 - \sqrt{x}, \quad x \geq 0$

(d) $\dfrac{1}{x^2+1}, \quad x \in \mathbb{R}$

7. Consider $f(x) = \sqrt{2-x}$.

(a) State the domain of f.

(b) State the range of f.

(c) Compare the domain of f with the domain of $g(x) = \sqrt{x-2}$.

Reading a graph

8. GDC Consider $f(x) = x^3 - x^2 - 4x + 2$. Give coordinates to three significant figures where they are not exact.

(a) Write down the y-intercept of the graph of f.

(b) Use a GDC to find the zeros of f.

(c) Use a GDC to find the coordinates of the local maximum point and the local minimum point.

Inverse functions

9. Find the inverse of each function.

(a) $f(x) = 3x - 6$

(b) $f(x) = \dfrac{x+1}{2}$

(c) $f(x) = x^3 + 1$

(d) $f(x) = \dfrac{1}{x+2}$

(e) $f(x) = \dfrac{2x+1}{x-3}$

(f) $f(x) = \dfrac{x-4}{x+2}$

10. Consider $f(x) = \dfrac{3x+2}{x-1}$.

(a) State the domain of f.

(b) Find $f^{-1}(x)$.

(c) State the domain of f^{-1}.

(d) Verify that $f^{-1}(f(2)) = 2$.

11. Use an inverse function to solve each equation.

(a) For $f(x) = 5x - 3$, find $f^{-1}(x)$, and hence solve $f(x) = 17$ by evaluating $f^{-1}(17)$.

(b) A function g has inverse $g^{-1}(x) = \dfrac{2x+1}{x-3}$, $x \neq 3$. Solve $g(x) = 5$ without finding $g(x)$.

12. The function $f(x) = x^2$ has no inverse over all real numbers. State a restricted domain on which an inverse does exist, and give that inverse.

Concept check

13. A student writes: "$f(x) = 2x + 3$, *so* $f^{-1}(x) = \dfrac{1}{2x+3}$." Identify the error and find the correct $f^{-1}(x)$.

4.2 Linear Functions

The simplest function that actually changes is the line: its output rises at a constant rate. That rate is the **gradient**, the rise divided by the run.

KEY · IB The gradient of the line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

A gradient and one point on the line are enough to fix the line completely. From that fact come three ways to write its equation. All three describe the same line.

KEY · IB A straight line can be written

$$y = mx + c, \qquad y - y_1 = m(x - x_1), \qquad ax + by + d = 0.$$

Here m is the gradient and c the y-intercept.

Use the first form when you know the gradient and the y-intercept. Use the second when you know the gradient and any point. The third is the general form, with whole-number coefficients.

Two lines are **parallel** when they rise at the same rate. They are **perpendicular** when you get one gradient from the other by turning the fraction upside down and changing the sign.

KEY For two lines with gradients m_1 and m_2:

$$m_1 = m_2 \ \text{(parallel)}, \qquad m_1 m_2 = -1 \ \text{(perpendicular)}.$$

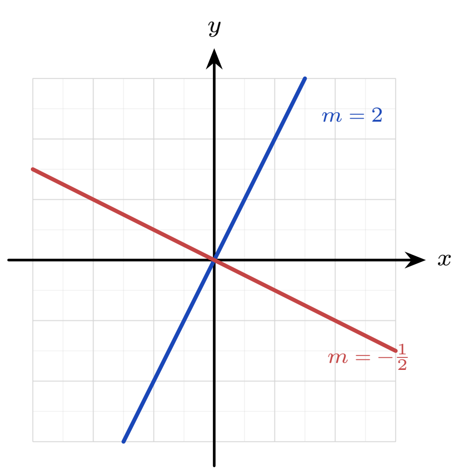


Figure 4.2 Perpendicular gradients: turn the fraction upside down and change the sign.

Worked through. To find the equation of the line through $(1, 2)$ and $(4, 8)$, first find the gradient:

$$m = \frac{8 - 2}{4 - 1} = \frac{6}{3} = 2.$$

Then use the point $(1, 2)$ in point–gradient form:

$$y - 2 = 2(x - 1) \implies y = 2x.$$

EXAMPLE 4.6

Find the equation of the line through $(0, 4)$ perpendicular to $y = 3x - 1$.
The given line has gradient 3, so the perpendicular gradient is $-\frac{1}{3}$. Through $(0, 4)$,

$$y = -\tfrac{1}{3}x + 4.$$

Two points fix more than a gradient. They also have a distance between them and a point halfway along, and both come from the coordinates directly. The distance is Pythagoras applied to the horizontal and vertical gaps; the midpoint is the average of the coordinates.

KEY · IB For the points (x_1, y_1) and (x_2, y_2), the distance between them and the midpoint of the segment joining them are

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}, \qquad M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The order of the points does not matter in either formula: the differences are squared, and the sums are symmetric. Both formulas extend to three dimensions by adding a z-term to each; see Ch. 7.

Together with a perpendicular gradient these give the **perpendicular bisector** of a segment, the line at right angles to it through its midpoint. Every point on the perpendicular bisector of AB is the same distance from A as from B, so it is also the set of all such points.

EXAMPLE 4.7

The points $A(2, -1)$ and $B(6, 7)$ are given. Find the length of AB, the midpoint of AB, and the equation of the perpendicular bisector of AB.
The horizontal gap is 4 and the vertical gap is 8, so

$$AB = \sqrt{(2 - 6)^2 + (-1 - 7)^2} = \sqrt{16 + 64} = \sqrt{80} = 4\sqrt{5}.$$

The midpoint is

$$M = \left(\frac{2 + 6}{2}, \frac{-1 + 7}{2}\right) = (4, 3).$$

The gradient of AB is $\dfrac{7 - (-1)}{6 - 2} = 2$, so the bisector has gradient $-\frac{1}{2}$. Through $M(4, 3)$,

$$y - 3 = -\tfrac{1}{2}(x - 4) \implies y = -\tfrac{1}{2}x + 5.$$

Every point of this line is the same distance from A as from B. Take $(0,5)$ on it: the distance to A is $\sqrt{4+36}=\sqrt{40}$, and to B it is $\sqrt{36+4}=\sqrt{40}$.

YOUR TURN 4.2 Linear Functions

Gradient and equation of a line

14. Find the gradient of the line through each pair of points.

(a) $(1,3)$ and $(5,11)$
(b) $(2,-1)$ and $(5,8)$
(c) $(-1,4)$ and $(3,-4)$
(d) $(0,5)$ and $(2,5)$

15. Find the equation of each line, giving your answer as $y=mx+c$.

(a) gradient 2, through $(0,-3)$
(b) through $(2,1)$ and $(4,7)$
(c) gradient $-\frac{1}{2}$, through $(4,0)$
(d) through $(2,-1)$, perpendicular to $y=-2x+3$

16. Write each equation as $y=mx+c$ and state the gradient.

(a) $2x+3y=6$
(b) $4x-2y=8$

17. Find the equation of each line in the form $ax+by+d=0$, where a, b and d are integers.

(a) gradient $\frac{2}{3}$, through $\left(\frac{1}{2},-1\right)$
(b) through $(-1,2)$ and $(3,-4)$

Parallel and perpendicular lines

18. Answer each question about parallel and perpendicular lines.

(a) Write down the gradient of a line perpendicular to $y=4x+1$.
(b) Write down the gradient of a line parallel to $3x+y=7$.
(c) Are $y=2x+1$ and $y=2x-3$ parallel? Give a reason.
(d) Are $y=3x-1$ and $y=-\frac{1}{3}x+2$ perpendicular? Give a reason.

Distance, midpoint and perpendicular bisector

19. For each pair of points, find the exact distance between them and the midpoint of the segment joining them.

(a) $(1, 2)$ and $(4, 6)$

(b) $(-3, 5)$ and $(1, -1)$

20. The points $A(-1, 4)$ and $B(5, 0)$ are given.

(a) Find the midpoint M of AB.

(b) Find the equation of the perpendicular bisector of AB.

(c) The point $P(k, 8)$ lies on the perpendicular bisector. Find k, and verify that $PA = PB$.

Concept check

21. A student writes: "*The line $2x + 3y = 6$ has gradient 2, so a line perpendicular to it has gradient $-\frac{1}{2}$.*" Identify the error and find the correct perpendicular gradient.

4.3 Systems of Linear Equations HL

Two lines in a plane usually cross at one point. Finding that point means finding the values of x and y that satisfy *both* equations at once, a **system** of equations. The standard tool is **elimination**: combine the equations to cancel one variable, solve for the other, then substitute back.

Worked through. To solve $2x + y = 7$ and $x - y = 2$, add the two equations, which cancels y:

$$(2x + y) + (x - y) = 7 + 2 \implies 3x = 9 \implies x = 3.$$

Substituting into $x - y = 2$ gives $3 - y = 2$, so $y = 1$. The lines cross at $(3, 1)$.

Not every system has a single answer, and the picture shows why. Two lines can do three things. They cross once, which gives a **unique solution**. They stay parallel and never meet, so there is **no solution**. Or they are the same line drawn twice, which gives **infinitely many solutions**.

CAUTION If two equations have the same gradient but different intercepts, such as $2x + y = 3$ and $4x + 2y = 10$, the lines are parallel and the system has *no* solution. Elimination then produces a false statement such as $0 = 4$.

The same idea extends to three equations in three unknowns, where each equation is now a plane in space. You eliminate one variable at a time to reduce a 3×3 system to a 2×2 one, or read the solution directly from a GDC.

EXAMPLE 4.8

Solve the system

$$x + y + z = 6, \qquad 2x - y + z = 3, \qquad x + 2y - z = 2.$$

Eliminate z in pairs. Subtracting the first equation from the second:

$$(2x - y + z) - (x + y + z) = 3 - 6 \implies x - 2y = -3.$$

Adding the first and third equations:

$$(x + 2y - z) + (x + y + z) = 2 + 6 \implies 2x + 3y = 8.$$

Now solve this 2×2 system. From $x - 2y = -3$ we get $x = 2y - 3$; substituting,

$$2(2y - 3) + 3y = 8 \implies 7y = 14 \implies y = 2,$$

so $x = 1$. Finally the first equation gives $1 + 2 + z = 6$, hence $z = 3$. The solution is $(x, y, z) = (1, 2, 3)$.

Eliminating one variable at a time, working down the system, is called **row reduction**. It is what your GDC's simultaneous-equation solver does inside. Use the solver to check your work. You still need the algebra in an examination, and you need it when the solver returns only an error message.

4.3.1 The augmented matrix

In the elimination above, what changed was never the letters x, y, z, only the numbers in front of them. The letters only marked the columns, so you can stop writing them. For a new system, the coefficients and constants alone form a grid called the **augmented matrix**:

$$\begin{cases} x + y + z = 6 \\ 2x - y + z = 7 \\ x + 2y - z = 3 \end{cases} \qquad \longrightarrow \qquad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & -1 & 3 \end{array}\right)$$

The vertical bar shows where the equals signs were. Nothing new is happening: the matrix records the same elimination in a shorter form, with fewer symbols to copy incorrectly.

Elimination becomes three **row operations**. None of them changes the solution set. Each one is reversible, and each does to a whole equation only what you were always allowed to do:

KEY The three elementary row operations:

$$R_i \leftrightarrow R_j \ \text{(swap two rows)}, \qquad R_i \to kR_i \ \text{(multiply a row by } k \neq 0\text{)},$$

$$R_i \to R_i + kR_j \ \text{(add a multiple of one row to another)}.$$

Swapping reorders the equations, scaling multiplies one through by a constant, and the third adds a multiple of one equation to another: the same three moves as elimination, written compactly.

The first non-zero entry of a row is its **pivot**. The goal is a step shape, with zeros below each pivot. The last row then contains one unknown, the row above it two, and so on. This shape is called **row echelon form**. To reach it, work one column at a time, using row operations to clear every entry below the pivot; a row swap that brings a 1 into the pivot position keeps the arithmetic simple. Once you reach it, work upwards from the last row, finding one unknown at a time. That is **back-substitution**.

Worked through. To solve the system displayed above by row reduction, work on its augmented matrix, clearing the first column below the pivot. $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & -1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -5 \\ 0 & 1 & -2 & -3 \end{array}\right)$$

A leading entry of 1 gives simpler arithmetic than -3, so swap $R_2 \leftrightarrow R_3$. Then clear below it with $R_3 \to R_3 + 3R_2$:

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -3 \\ 0 & -3 & -1 & -5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -7 & -14 \end{array}\right)$$

The step shape is complete. Now read it from the bottom up. The last row says $-7z = -14$, so $z = 2$. The middle row says $y - 2z = -3$, so $y = 4 - 3 = 1$. The top row gives $x = 6 - y - z = 3$. The solution is $(x, y, z) = (3, 1, 2)$.

Every step here is the elimination you already know. The matrix keeps the working short and the columns lined up, and misaligned columns are a common cause of mistakes.

Once a system is in row echelon form, its rows determine the outcome. There are only three possible outcomes, and every 3×3 system gives exactly one of them.

Rows reduce to	What it means	The three planes
last row $(0\ 0\ a \mid b)$, $a \neq 0$	one unique solution	meet at a single point
any row $(0\ 0\ 0 \mid c)$, $c \neq 0$	no solution: the row says $0 = c$, which is false	have no common point; two are parallel, or the three meet in a triangular prism
$(0\ 0\ 0 \mid 0)$, and no row of the type above	infinitely many solutions: the row says $0 = 0$, which is true but tells us nothing	share a whole line, or a whole plane if two rows vanish

First look for a row $(0\ 0\ 0 \mid c)$ with $c \neq 0$. Any such row, in any position, means there is no solution. If there is none, count the rows that become all zeros. No zero row gives one solution. One zero row gives a line of solutions, described by one free variable. Two zero rows mean all three equations describe the same plane, and the solutions form that whole plane, described by two free variables.

4.3.2 When a system has no unique solution

The table above lists the three outcomes. When one row reduces to $0 = 0$, choose an unknown in the last non-zero row as the free variable and set it equal to a parameter, such as $z = t$. Back-substitute up the echelon form to write each other unknown in terms of t. The result, (x, y, z) in terms of t with $t \in \mathbb{R}$, is the **general solution**, and it describes the line where the planes meet. In the system worked through below, a constant k in the equations decides between the outcomes.

Worked through. To find the value of k for which the system

$$x + y + 2z = 2, \qquad 2x - y + z = 1, \qquad x + 4y + 5z = k$$

has infinitely many solutions, give the general solution, and show that every other value of k gives no solution, look for a row that reduces to zeros. Eliminate x using the first equation. Second minus twice the first:

$$(2x - y + z) - 2(x + y + 2z) = 1 - 4 \implies -3y - 3z = -3 \implies y + z = 1.$$

Third minus the first:

$$(x + 4y + 5z) - (x + y + 2z) = k - 2 \implies 3y + 3z = k - 2 \implies y + z = \frac{k-2}{3}.$$

The same left side, $y + z$, has appeared twice. The two remaining equations are consistent only if

$$1 = \frac{k-2}{3} \implies k = 5.$$

If $k \neq 5$, subtracting them gives $0 = \frac{k-5}{3} \neq 0$, a false statement: the system is inconsistent, and no solution exists for any such k.

If $k = 5$, the third equation gives no new information, and the system reduces to two equations. Let the free variable be $z = t$. Then $y = 1 - t$, and the first equation gives

$$x = 2 - y - 2z = 2 - (1 - t) - 2t = 1 - t.$$

The general solution is

$$(x, y, z) = (1 - t,\ 1 - t,\ t), \qquad t \in \mathbb{R},$$

a whole line of solutions: the line where the three planes meet. Check with $t = 0$: the point $(1, 1, 0)$ satisfies all three original equations. So does $(0, 0, 1)$, from $t = 1$. One free parameter gives one line, so the geometry and the algebra give the same result.

CAUTION "No unique solution" covers two different cases, so always separate them. A row that reduces to $0 = 0$ gives infinitely many solutions. A row that reduces to $0 = c$, with $c \neq 0$, gives none. Examination questions can ask for both, and for the general solution in terms of a parameter.

YOUR TURN 4.3 Systems of Linear Equations HL

Two equations in two unknowns

22. Solve each system by elimination.

(a) $x + y = 5, \quad x - y = 1$
(b) $3x - y = 5, \quad x + y = 3$
(c) $x + y = 4, \quad 2x + y = 7$
(d) $2x + 3y = 12, \quad x - y = 1$

23. Solve each system by substitution.

(a) $y = 2x, \quad x + y = 6$
(b) $y = x - 1, \quad 2x + y = 8$

24. State how many solutions each system has, and give a reason.

(a) $2x + y = 4, \quad 4x + 2y = 9$
(b) $y = x + 1, \quad y = x + 3$
(c) $x + y = 3, \quad 2x + 2y = 6$
(d) $x + y = 3, \quad x - y = 1$

Three equations in three unknowns

25. Solve each system of three equations.

(a) $x+y+z=6, \quad x-y+z=2, \quad x+y-z=0$

(b) $x+y+z=4, \quad 2x-y+z=8, \quad x+2y-z=-3$

(c) $x+y+z=2, \quad 2x-y+3z=-1, \quad x-2y+z=-1$

(d) $2x+y-z=-4, \quad x+3y+2z=3, \quad 3x-y+z=9$

Row reduction and the number of solutions

26. For parts (a) to (c), write the augmented matrix, reduce it to row echelon form, and state the number of solutions.

(a) $x+y+z=6, \quad x+2y-z=2, \quad 2x+3y+z=11$

(b) $x+y+z=3, \quad 2x+2y+2z=7, \quad x-y+z=1$

(c) $x+y+z=6, \quad x-y+z=2, \quad 2x+2z=8$

(d) For $x+y+z=3, \quad 2x+y+3z=7, \quad 3x+2y+4z=k$, find the value of k for which the system has infinitely many solutions, and state what happens for every other value of k.

Concept check

27. A student eliminates x from $x+2y=4$ and $2x+4y=8$, obtains $0=0$, and writes "*no solution*".

(a) Explain why this conclusion is wrong, and state the correct number of solutions.

(b) The second equation is changed to $2x+4y=10$. State what elimination now gives, and what that result means.

4.4 Quadratic Functions

A **quadratic** is a function whose highest power is x^2. Its graph is the **parabola**, the curve of Galileo's cannonball: a single smooth arch, symmetric about a vertical line through its turning point.

The same quadratic can be written in three forms, and each form shows a different feature immediately.

KEY A quadratic can be written in three forms:

$$y = ax^2 + bx + c, \qquad y = a(x-p)(x-q), \qquad y = a(x-h)^2 + k.$$

Expanded form shows the y-intercept c; **factorised** form shows the roots p and q; **vertex** form shows the turning point (h, k).

The sign of a sets the direction. If $a > 0$ the parabola opens upward and the turning point is a minimum. If $a < 0$ it opens downward and the turning point is a maximum. The curve is symmetric, and its mirror line passes through the vertex.

KEY · IB The graph of $f(x) = ax^2 + bx + c$ has axis of symmetry

$$x = -\frac{b}{2a}.$$

To move from expanded form to vertex form, **complete the square**. This means rewriting $x^2 + \frac{b}{a}x$ as a perfect square, then subtracting whatever that added. It is used throughout the course.

Worked through. To write $y = x^2 - 10x + 21$ in vertex form, take the $x^2 - 10x$ part and complete the square. Half of -10 is -5, and $(-5)^2 = 25$:

$$x^2 - 10x = (x-5)^2 - 25.$$

So

$$y = (x-5)^2 - 25 + 21 = (x-5)^2 - 4.$$

The vertex is $(5, -4)$ and the axis of symmetry is $x = 5$. Because $a = 1 > 0$, this vertex is the minimum point.

EXAMPLE 4.9

Sketch $y = x^2 - 2x - 3$, showing the vertex, axis of symmetry, and all intercepts.

Factorise for the roots: $x^2 - 2x - 3 = (x-3)(x+1)$, so the x-intercepts are $x = -1$ and $x = 3$. The y-intercept is $c = -3$. Completing the square, $y = (x-1)^2 - 4$, so the vertex is $(1, -4)$ and the axis of symmetry is $x = 1$, which is exactly halfway between the two roots.

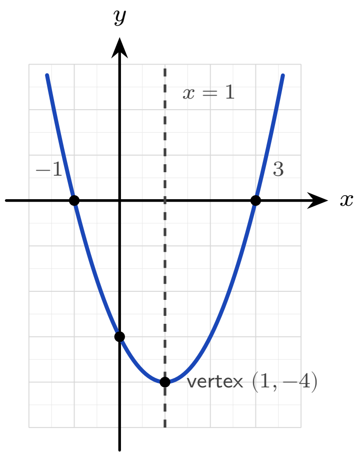


Figure 4.3 $y = x^2 - 2x - 3$, with its vertex, axis of symmetry and intercepts.

The three forms also work in reverse. Given points on a parabola, choose the form that already contains what you know, then use the remaining point to fix a. Two roots and one further point call for the factorised form.

EXAMPLE 4.10

A parabola passes through $(0, -8)$, $(-1, 0)$ and $(4, 0)$. Find its equation.

Two of the points lie on the x-axis, so they are the roots. The factorised form gives

$$y = a(x+1)(x-4),$$

with a still unknown: infinitely many parabolas share a pair of roots, and they differ only in how steeply they open. Substitute the third point, $(0, -8)$, to pin it down:

$$-8 = a(0+1)(0-4) = -4a \implies a = 2.$$

Expanding,

$$y = 2(x+1)(x-4) = 2x^2 - 6x - 8.$$

Check the three points: at $x = 0$, $y = -8$; at $x = -1$ and $x = 4$, $y = 0$. Had the curve passed through $(0, -4)$ instead, the same substitution would give $a = 1$, halving every y-coordinate while leaving the roots where they are.

YOUR TURN 4.4 Quadratic Functions

Completing the square

28. Write each quadratic in vertex form and state its vertex.

(a) $y = x^2 + 4x + 1$ (b) $y = x^2 - 8x + 10$
(c) $y = x^2 + 3x + 1$ (d) $y = x^2 + 2x + 5$

29. Write each quadratic in the form $a(x-h)^2 + k$. *(Hint: take out the factor a from the x^2 and x terms first)*

(a) $2x^2 + 12x + 5$ (b) $3x^2 - 6x + 1$
(c) $-x^2 + 6x - 4$ (d) $-2x^2 - 8x + 3$

Vertex, axis and range

30. State the vertex and the axis of symmetry of each quadratic.

(a) $y = (x-2)^2 + 5$ (b) $y = x^2 - 8x + 3$
(c) $y = -(x+1)^2 + 2$ (d) $y = 2(x-3)^2 - 1$

31. Find the range of each quadratic by completing the square. Each is defined for all real x.

(a) $x^2 - 6x + 11$ (b) $x^2 + 5x$
(c) $2x^2 - 4x + 7$ (d) $1 + 4x - 2x^2$

32. State whether each quadratic has a maximum or a minimum, give a reason, and find that maximum or minimum value.

(a) $y = -x^2 + 2x$ (b) $y = 3x^2 - x + 1$
(c) $y = x^2 - 6x + 5$ (d) $y = 5 + 4x - x^2$

Roots, intercepts and sketches

33. Consider $y = (x-1)(x+5)$.

(a) Write down its roots. (b) Find its y-intercept.
(c) Find its axis of symmetry. (d) Find its vertex.

34. Sketch each parabola, showing the vertex, the axis of symmetry and all intercepts.

(a) $y = -x^2 + 2x + 8$ (b) $y = 2x^2 - 4x - 6$

Finding the equation of a parabola

35. Find the equation of each parabola in the form $y = ax^2 + bx + c$.

(a) It has roots $x = -2$ and $x = 3$, and passes through $(0, 12)$.
(b) It passes through $(1, 0)$, $(5, 0)$ and $(3, -8)$.
(c) Its vertex is $(2, -3)$, and it passes through $(0, 5)$.

Concept check

36. A student states that the vertex of $y = (x+3)^2 - 5$ is $(3, -5)$. Identify the error and state the correct vertex.

37. A student completes the square as follows.

$$3x^2 - 12x + 5 = 3(x-2)^2 - 4 + 5 = 3(x-2)^2 + 1$$

Identify the error, correct the working, and state the vertex.

4.5 Quadratic Equations

Setting a quadratic equal to zero is the same as asking where its parabola crosses the x-axis. There are three ways to find those crossings, and all three give the same answers. Choosing the right method saves time in an examination.

Method	Use it when	Be careful with
Factorisation	the quadratic factorises with whole numbers	many quadratics do not factorise, and searching for factors takes time
Completing the square	you also want the vertex, or an exact surd answer	the arithmetic is harder when $a \neq 1$
The formula	always. It works for every quadratic	slower than factorising when factors are obvious

4.5.1 Solving by factorisation

If the quadratic factorises, you can read the roots immediately. The reason is the zero product rule: a product is zero only when one of its factors is zero.

Worked through. To solve $x^2 - 8x + 15 = 0$, find two numbers that multiply to 15 and add to -8. They are -3 and -5:

$$(x-3)(x-5) = 0.$$

A product is zero only when a factor is zero, so $x - 3 = 0$ or $x - 5 = 0$, giving $x = 3$ or $x = 5$.

CAUTION The zero product rule needs a zero on the right. From $(x-2)(x-3) = 2$ you may *not* conclude $x - 2 = 2$. Expand, collect everything on one side, and factorise again.

4.5.2 Solving by completing the square

Completing the square works whether or not the quadratic factorises, and it gives exact answers in surd form. Rewrite the quadratic as a perfect square plus a constant, then take square roots of both sides.

Worked through. To solve $x^2 + 4x - 12 = 0$ by completing the square, note that half of 4 is 2, and $2^2 = 4$, so $x^2 + 4x = (x+2)^2 - 4$. The equation becomes

$$(x+2)^2 - 4 - 12 = 0 \implies (x+2)^2 = 16.$$

Take the square root of both sides, remembering both signs:

$$x + 2 = \pm 4 \implies x = 2 \text{ or } x = -6.$$

EXAMPLE 4.11

Solve $x^2 - 4x - 1 = 0$, giving exact answers.
Half of -4 is -2, and $(-2)^2 = 4$:

$$(x-2)^2 - 4 - 1 = 0 \implies (x-2)^2 = 5 \implies x - 2 = \pm\sqrt{5}.$$

So $x = 2 \pm \sqrt{5}$. This quadratic does not factorise with whole numbers, which is why the answer contains a surd.

TIP When $a \neq 1$, divide the whole equation by a first. For $2x^2 - 8x + 5 = 0$, divide by 2 to get $x^2 - 4x + 2.5 = 0$, then complete the square as usual.

4.5.3 Solving by formula

The formula works for every quadratic, and it is in the formula booklet.

KEY · IB The solutions of $ax^2 + bx + c = 0$, with $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Substitute the three coefficients and the formula returns both roots at once.

The formula is not a rule to memorise without understanding. It is what **completing the square** gives when the coefficients are letters instead of numbers, so you can derive it yourself from $ax^2 + bx + c = 0$ at any time.

Divide through by a, then complete the square on $x^2 + \frac{b}{a}x$:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \implies \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0.$$

Move the two constants to the right and write them over the common denominator $4a^2$:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Take the square root of both sides, which is where the $\pm$ enters, and then subtract $\frac{b}{2a}$:

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The $\pm$ is the algebra recording what the picture already shows: the parabola is symmetric, with one root on each side of its axis. The expression under the root, $b^2 - 4ac$, decides how many roots there are, and that is the subject of the next section.

Worked through. To solve $2x^2 - 5x - 12 = 0$ by the formula, identify the coefficients: $a = 2$, $b = -5$, $c = -12$. The formula gives

$$x = \frac{5 \pm \sqrt{25 + 96}}{4} = \frac{5 \pm 11}{4},$$

so $x = 4$ or $x = -\frac{3}{2}$.

YOUR TURN 4.5 Quadratic Equations

Factorisation

38. Solve each equation by factorisation.

(a) $x^2 - 5x + 6 = 0$

(b) $x^2 - x - 12 = 0$

(c) $x^2 + x - 20 = 0$

(d) $2x^2 - 5x - 3 = 0$

Rearranging to zero first

39. Rearrange each equation so that one side is 0, then solve it by factorisation.

(a) $x^2 = 7x$

(b) $x(x + 4) = 21$

(c) $(x - 2)(x - 3) = 2$

(d) $3x^2 = 2(x + 4)$

Completing the square

40. Solve each equation by completing the square, giving exact answers.

(a) $x^2 + 8x + 7 = 0$

(b) $x^2 - 2x - 5 = 0$

(c) $x^2 + 6x + 2 = 0$

(d) $2x^2 + 12x + 7 = 0$

The quadratic formula

41. Solve each equation with the quadratic formula, giving exact answers.

(a) $x^2 + 2x - 1 = 0$
(b) $x^2 + 6x + 4 = 0$
(c) $2x^2 - 3x - 2 = 0$
(d) $3x^2 + 2x - 1 = 0$

Choosing a method

42. For each equation, state which of the three methods you would choose, and give a reason. You do not need to solve the equations.

(a) $x^2 - 7x + 12 = 0$
(b) $x^2 + 4x - 2 = 0$
(c) $5x^2 - 3x - 1 = 0$
(d) $x^2 - 9 = 0$

Concept check

43. A student solves $x^2 = 5x$ by dividing both sides by x, and writes "$x = 5$". Identify the error and give the complete solution.

4.6 The Discriminant

The quadratic formula gives more information than the roots. Before you calculate a single root, the quantity under the square root sign already tells you how many real roots there are. That quantity has its own name.

KEY · IB The **discriminant** of $ax^2 + bx + c$ is

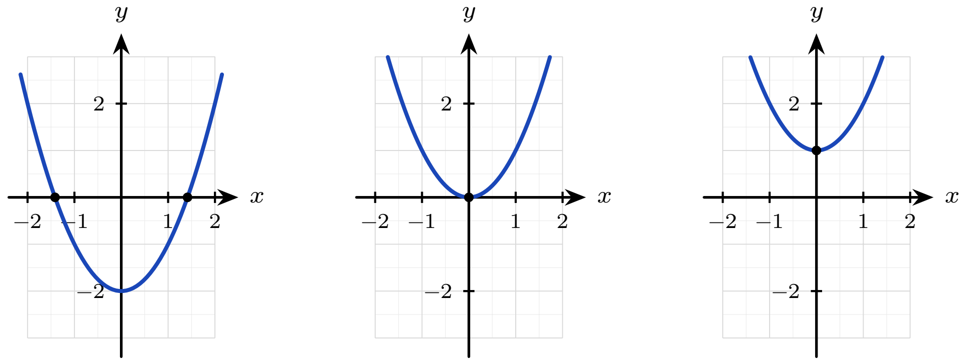
$$\Delta = b^2 - 4ac.$$

Its sign tells you how many real roots the equation has, without solving it.

The sign of Δ determines how many times the parabola meets the x-axis. The reason is that a negative number has no real square root, so $\pm\sqrt{\Delta}$ has no real value.

Discriminant	Real roots	Parabola and x-axis
$\Delta > 0$	two distinct	crosses twice
$\Delta = 0$	one repeated	touches once
$\Delta < 0$	none	does not meet it

The three cases are three positions of the same parabola relative to the x-axis.



(a) $\Delta > 0$: crosses twice (b) $\Delta = 0$: touches once (c) $\Delta < 0$: never meets

Figure 4.4 The same upward parabola in three positions: $y = x^2 - 2$, $y = x^2$ and $y = x^2 + 1$. Lowering it past the axis gives two roots, resting it on the axis gives one, and lifting it clear gives none. The discriminant measures which of the three has happened.

A downward parabola behaves the same way, with the picture turned over: the sign of a decides which way it opens, and Δ still counts the crossings.

Worked through. For $x^2 + kx + 9 = 0$ to have exactly one (repeated) root, the discriminant must be zero, $\Delta = 0$:

$$k^2 - 4(1)(9) = 0 \implies k^2 = 36 \implies k = \pm 6.$$

At these two values the parabola touches the x-axis at one point. Notice that the discriminant answers a question about the roots without finding them.

EXAMPLE 4.12

For the equation $3kx^2 + 2x + k = 0$, find the values of k that give two distinct real roots, two equal real roots, and no real roots.

Here the unknown constant appears in two places: $a = 3k$ and $c = k$, with $b = 2$. So

$$\Delta = b^2 - 4ac = 4 - 4(3k)(k) = 4 - 12k^2.$$

Now consider the three cases separately.

$$\Delta > 0: \ 4 - 12k^2 > 0 \implies k^2 < \tfrac{1}{3} \implies -\tfrac{1}{\sqrt{3}} < k < \tfrac{1}{\sqrt{3}}.$$

$$\Delta = 0: \ k^2 = \tfrac{1}{3} \implies k = \pm\tfrac{1}{\sqrt{3}}.$$

$$\Delta < 0: \ k^2 > \tfrac{1}{3} \implies k < -\tfrac{1}{\sqrt{3}} \ \text{ or } \ k > \tfrac{1}{\sqrt{3}}.$$

One value of k must be checked separately. If $k = 0$ the equation becomes $2x = 0$. That is linear, not quadratic, and it has the single root $x = 0$. So the answer for two distinct real roots is $-\frac{1}{\sqrt{3}} < k < \frac{1}{\sqrt{3}}$, *with* $k \neq 0$.

CAUTION Before using the discriminant, check that the equation really is quadratic. If the coefficient of x^2 contains the unknown constant, the value that makes it zero must be excluded and treated separately.

TIP When the equation is a quadratic, a question that says "two distinct roots", "equal roots", "a repeated root", "touches the x-axis", or "no real solutions" usually points to the discriminant. Recognising the wording tells you which method to use.

4.6.1 A line meeting a parabola

The discriminant does more than count where a parabola meets the x-axis. A point common to two graphs satisfies both equations, so setting the two expressions for y equal gives a quadratic whose real roots are the x-coordinates of the intersections. The discriminant counts them. When there is exactly one, the line touches the curve without crossing it, and is a **tangent**.

Worked through. To read the three cases geometrically, substitute $y = mx + c$ into $y = x^2 - 2x + 5$; then, to find the values of m for which $y = mx + 4$ is a tangent to that parabola, take a particular value of c. At any intersection both equations hold, so

$$x^2 - 2x + 5 = mx + c \implies x^2 - (m+2)x + (5-c) = 0.$$

Its coefficients are 1, $-(m+2)$ and $5 - c$, so

$$\Delta = (m+2)^2 - 4(1)(5-c) = (m+2)^2 + 4c - 20.$$

The constant term is $5 - c$, not c; the intercept c changes sign when it moves to the left-hand side. Each real root is one intersection point, so $\Delta > 0$ gives two intersections, $\Delta = 0$ gives one and the line is a tangent, and $\Delta < 0$ means the line misses the parabola entirely.

For $y = mx + 4$ take $c = 4$ and impose tangency:

$$\Delta = (m+2)^2 - 4 = 0 \implies m + 2 = \pm 2 \implies m = 0 \text{ or } m = -4.$$

Check $m = 0$: the quadratic is $x^2 - 2x + 1 = (x-1)^2$, a repeated root at $x = 1$, so the horizontal line $y = 4$ touches at the vertex $(1, 4)$. For $m = -4$ the quadratic is $x^2 + 2x + 1 = (x+1)^2$, touching at $(-1, 8)$.

A tangent shows up as a repeated root, so factorising the quadratic at each value of m both confirms the answer and gives the point of contact.

4.6.2 The sum and product of the roots HL

The roots of a quadratic are linked to its coefficients before you solve anything. If $ax^2 + bx + c$ has roots α and β, with $a \neq 0$ so that the expression really is quadratic, then it factorises as

$$a(x-\alpha)(x-\beta) = ax^2 - a(\alpha+\beta)x + a\,\alpha\beta.$$

Comparing the coefficient of x gives $b = -a(\alpha+\beta)$, and comparing the constant term gives $c = a\,\alpha\beta$. Dividing each by a, which the condition $a \neq 0$ permits, leaves the sum of the roots as $-\frac{b}{a}$ and their product as $\frac{c}{a}$. Nothing in that expansion depended on there being two brackets, so the same comparison settles a polynomial of any degree, and the result is stated properly in Ch. 6.

KEY HL If $ax^2+bx+c=0$ has roots α and β, then

$$\alpha+\beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{c}{a}.$$

The booklet does not print this form. It prints the rule for any degree (booklet 2.12, Ch. 6), and this is its case $n = 2$.

These let you find expressions in the roots without solving the equation. Any expression that stays the same when α and β are swapped can be rewritten in terms of $\alpha+\beta$ and $\alpha\beta$; for example, $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$.

Worked through. To find $\alpha^2+\beta^2$, where α and β are the roots of $2x^2-6x+3=0$, without solving the equation, read the sum and product from the coefficients. Here $\alpha+\beta = -\frac{-6}{2} = 3$ and $\alpha\beta = \frac{3}{2}$. Since $(\alpha+\beta)^2 = \alpha^2+2\alpha\beta+\beta^2$,

$$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = 9-3 = 6.$$

The identity for $\alpha^2+\beta^2$ did the work, and the equation was never solved.

YOUR TURN 4.6 The Discriminant

The nature of the roots

44. Evaluate the discriminant of each equation and state the nature of its roots.

(a) $x^2-4x+4=0$

(b) $2x^2+3x+5=0$

(c) $3x^2-2x-1=0$

(d) $x^2+5x+7=0$

Conditions on a coefficient

45. Find the value or values of k in each case.

(a) $x^2 + 6x + k = 0$ has equal roots
(b) $x^2 + (k-2)x + 4 = 0$ has equal roots
(c) $x^2 - 3x + k = 0$ has no real roots
(d) $(k-1)x^2 + 4x + 2 = 0$ has two distinct real roots. State any value of k that must be excluded.

46. The equation $x^2 + (k+1)x + 9 = 0$ has two distinct real roots. Find the set of values of k.

A line meeting a parabola

47. Each part concerns a line and a parabola.

(a) Find the number of points at which the line $y = x + 3$ meets the parabola $y = x^2 - 3x + 5$.
(b) Show that the line $y = 2x - 1$ is a tangent to $y = x^2 - 4x + 8$, and find the point of contact.
(c) Find the value of c for which the line $y = 3x + c$ is a tangent to $y = x^2 + x + 2$.
(d) Find the set of values of m for which the line $y = mx - 1$ does not meet $y = x^2 + 3$.

The sum and product of the roots

48. Answer each part without solving the equation.

(a) The roots of $3x^2 + 6x - 2 = 0$ are α and β. Write down $\alpha + \beta$ and $\alpha\beta$.
(b) The roots of $x^2 - 4x + 1 = 0$ are α and β. Find $\alpha^2 + \beta^2$.
(c) For the same equation as in part (b), find $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$.
(d) The roots of $2x^2 - 3x - 1 = 0$ are α and β. Find $(\alpha - \beta)^2$. *(Hint: expand $(\alpha-\beta)^2$ and compare it with $(\alpha+\beta)^2$)*

Concept check

49. A student says: "*the discriminant of $2x^2 - 3x + 5$ is negative, so the graph of $y = 2x^2 - 3x + 5$ lies below the x-axis.*" Explain why the conclusion is false.

50. Explain the difference between "*the equation has real roots*" and "*the equation has two distinct real roots*", and state the condition on the discriminant for each.

4.7 Quadratic Inequalities

A quadratic **inequality** is a different question. It does not ask where the parabola meets the axis. It asks where the parabola lies above the axis, or below it. The method has three steps.

KEY To solve a quadratic inequality:

1. Collect every term on one side, so the other side is 0.
2. Find the roots of the matching equation. These are the only places where the sign can change.
3. Sketch the parabola and read the intervals where it lies on the side you want.

With $<$ or $>$ the roots themselves are excluded; with $\leq$ or $\geq$ they are included, because the quadratic is zero there.

The sketch is the method, so here it is in full. The roots cut the axis into three stretches, and the curve keeps one sign right across each of them.

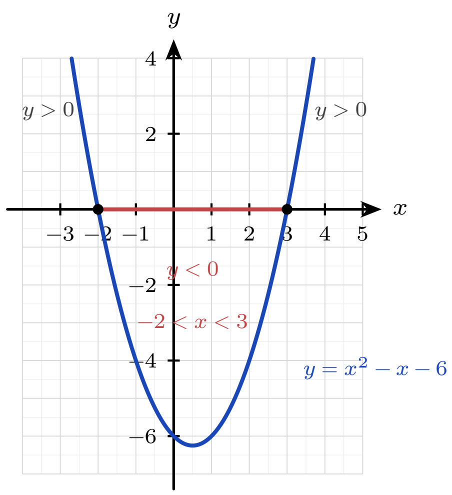


Figure 4.5 $y = x^2 - x - 6$, with roots at $x = -2$ and $x = 3$. The curve lies below the axis on the scarlet stretch between the roots and above it on both sides. So $x^2 - x - 6 < 0$ gives $-2 < x < 3$, and $x^2 - x - 6 > 0$ gives $x < -2$ or $x > 3$.

Read the picture, not a rule. An upward parabola is negative between its roots and positive outside them; a downward one is the other way round. A quick sketch helps prevent a sign error.

If the discriminant is negative there are no roots, so the parabola lies entirely on one side of the axis. The inequality then holds for every real x or for none, and the sign of a decides which.

EXAMPLE 4.13

Solve $x^2 - x - 6 > 0$.

Factorise: $x^2 - x - 6 = (x-3)(x+2)$, so the roots are $x = -2$ and $x = 3$. The parabola opens upward. It is therefore *above* the axis outside the roots, and below the axis between them. So

$$x^2 - x - 6 > 0 \quad \text{when} \quad x < -2 \text{ or } x > 3.$$

A rough sketch of the parabola is a reliable check on the signs.

EXAMPLE 4.14

Solve $3x^2 \leq 5x + 2$.

First collect everything on one side:

$$3x^2 - 5x - 2 \leq 0.$$

Factorise: $(3x+1)(x-2) = 0$ gives roots $x = -\frac{1}{3}$ and $x = 2$. The parabola opens upward, so it lies *below* the axis between the roots. The sign $\leq$ includes the roots themselves:

$$-\tfrac{1}{3} \leq x \leq 2.$$

CAUTION Do not divide an inequality by a term containing x unless its sign is known. Dividing by a negative quantity reverses the inequality. Collect on one side and factorise instead.

EXAMPLE 4.15

Solve $x^2 + 3x + 4 > 0$.

The discriminant is $9 - 16 = -7 < 0$, so the parabola never meets the x-axis. Since $a = 1 > 0$ it opens upward, so it lies entirely above the axis. The inequality holds for *every* real x.

This is the negative-discriminant case described above, with $a > 0$.

YOUR TURN 4.7 Quadratic Inequalities

51. Solve each inequality.

(a) $x^2 - 9 < 0$

(b) $x^2 + 2x - 15 < 0$

(c) $x^2 - 4x + 3 \geq 0$

(d) $-x^2 + 4x > 0$

52. Solve each inequality. Two of them are special cases, so justify your answers.

(a) $2x^2 \leq x + 6$

(b) $x^2 \geq 16$

(c) $x^2 + x + 1 > 0$

(d) $x^2 + 4 < 0$

53. A rectangle has width x and length $x + 2$, both in centimetres.

(a) Show that its area is $x^2 + 2x$ square centimetres.

(b) Find the values of x for which the area is less than 15 square centimetres.

Concept check

54. A student solves $x^2 > 9$ by writing "$x > \pm 3$, *so* $x > 3$ *or* $x > -3$." Identify the error and give the correct solution.

Chapter 4 · Lines & Quadratics

Reflections

TOK Descartes showed that a geometric problem can be solved with algebra, and an algebraic one with a picture. A line is an equation, and a linear equation is a line. Neither form is more real than the other. Does changing the representation give you new knowledge, or only a new view of knowledge you already had?

IM Around 1800 BCE, scribes in Babylon solved problems such as "the area of a square added to its side is $\frac{3}{4}$; find the side" by a fixed sequence of steps that is completing the square, written on clay tablets with no symbols at all. In 628 CE the Indian mathematician Brahmagupta stated a general rule for such equations in words, and gave rules for calculating with negative numbers, which he called debts, and with zero. The method of §4.4 is nearly four thousand years old; only the notation is new. Does a method belong to whoever first used it, or to whoever first wrote it down in general?

RW A gradient is a rate, so any line drawn between two quantities has one: a road sign showing 12%, a wheelchair ramp of about 1 in 12 in many building codes, speed on a distance-time graph, and marginal cost (the extra cost of making one more item) on a total-cost graph. The parabola has uses of its own. It reflects every incoming ray parallel to its axis of symmetry to a single point, the focus, so a satellite dish collects weak signals there, and a headlight with its bulb at the focus sends out a parallel beam. Quadratics also model optimisation problems, such as the greatest area for a fixed length of fence or the greatest height of a thrown ball.

EXT The discriminant divides quadratics into three cases. When $\Delta < 0$ the parabola does not meet the x-axis and there are no *real* roots, but the quadratic formula still returns answers, now containing $\sqrt{-1}$. Keeping these "impossible" numbers instead of rejecting them leads to the **complex numbers** of Chapter 17, where every quadratic has exactly two roots and the discriminant only tells you whether those roots are real.

End-of-Chapter Problems
Chapter 4: Lines & Quadratics

1. Consider $f(x) = \dfrac{3x-2}{x+1}$, $x \neq -1$.

(a) Find $f^{-1}(x)$. [3]

(b) State the domain of f^{-1}, and hence write down the range of f. [2]

2. Find the equation of the line through $(3, -1)$ parallel to $2x - y = 4$. [3]

3. The points $A(-2, 7)$ and $B(4, -1)$ lie on the line L_1.

(a) Find the gradient of L_1 and the distance AB. [3]

(b) Show that L_1 has equation $4x + 3y = 13$. [2]

(c) Write down the x- and y-intercepts of L_1. [2]

(d) The line L_2 has equation $x + y = 4$. Find the point of intersection of L_1 and L_2. [3]

(e) Show that this point is the midpoint of AB. [1]

4. A line passes through $A(1, 2)$ and $B(5, 4)$.

(a) Find the gradient of AB. [1]

(b) Find the equation of AB. [2]

(c) Find the equation of the perpendicular bisector of AB. [4]

5. Solve the system

$$x + 2y + z = 4, \qquad 2x + y - z = 2, \qquad x - y + 2z = 2.$$

[6]

6. Consider the system

$$x + y + z = 3, \qquad x + 2y + 3z = 4, \qquad 2x + 3y + 4z = k.$$

(a) Find the value of k for which the system has solutions. [3]

(b) For this value of k, find the general solution. [3]

(c) Describe the configuration of the three planes when $k = 10$. [1]

7. GDC A theatre sells adult tickets at \$14.50, student tickets at \$9.75 and child tickets at \$6.25. On one evening it sold 412 tickets for a total of \$4522. The number of adult tickets sold was 52 fewer than the number of student and child tickets together.

(a) Write down a system of three equations for the numbers a, s and c of adult, student and child tickets sold. [3]

(b) Use your GDC to solve the system. [2]

8. A quadratic function f has zeros at $x = -1$ and $x = 4$, and $f(0) = -8$.

(a) Write $f(x)$ in factorised form. [2]

(b) Write $f(x)$ in expanded form. [1]

(c) Find the coordinates of the vertex of the graph of f. [2]

(d) Hence write down the range of f. [1]

9. Consider $f(x) = x^2 + 2x - 3$.

(a) Factorise $f(x)$ and state its roots. [2]

(b) Find the vertex. [2]

(c) Sketch the graph, showing all intercepts. [2]

10. Consider $f(x) = -x^2 + 4x + 5$, $x \in \mathbb{R}$.

(a) Write $f(x)$ in the form $a(x-h)^2 + k$. [2]

(b) Write down the vertex and the axis of symmetry of the graph of f. [1]

(c) State the range of f. [1]

(d) Find the x- and y-intercepts of the graph of f. [2]

11. Solve the inequality $x^2 - 2x - 8 \leq 0$. [3]

12. Solve the inequality $x^2 + x - 6 > 0$. [3]

13. The equation $x^2 - 4x + c = 0$ has no real roots. Find the set of values of c. [3]

14. The equation $x^2 + kx + 4 = 0$ has two distinct real roots. Find the set of values of k. [4]

15. Find the values of k for which $x^2 + kx + (k+3) = 0$ has equal roots. [4]

16. Two numbers have a sum of 10 and a product of 21. Find the numbers. [3]

17. The roots of $x^2 + bx + c = 0$ sum to 6 and differ by 4. Find b and c. [3]

18. Show that $x^2 + (k+1)x + k = 0$ has real roots for every real k, and find the value of k for which the roots are equal. [4]

19. The roots of $x^2 - px + q = 0$ are α and 2α. Show that $2p^2 = 9q$. [4]

20. Find the points of intersection of $y = x^2$ and $y = 2x + 3$. [4]

21. Consider the line $y = 2x - 5$ and the parabola $y = x^2 - 2x + 2$.

(a) Show that they do not intersect. [3]
(b) Explain what this means geometrically. [1]

22. A rectangle has length 3 more than its width w.

(a) Write its area A as a function of w. [2]
(b) Given the area is 40, find w. [3]

23. A rectangle has perimeter 20. Show that its area is $A = x(10 - x)$, where x is one side, and find the maximum possible area. [5]

24. Consider the parabola $y = x^2$.

(a) Find the value of k for which the line $y = 3x + k$ is a tangent to the parabola. [2]
(b) Prove that the line $y = mx + c$ is a tangent to the parabola if and only if $c = -\dfrac{m^2}{4}$. [3]

25. GDC A ball is thrown upwards from a height of 1.6 m. Its height in metres t seconds later is $h(t) = 1.6 + 12.5t - 4.9t^2$, until it hits the ground.

(a) Find the maximum height of the ball, and the time at which it is reached. [2]
(b) Find the time at which the ball hits the ground. [2]
(c) Find the length of time for which the ball is more than 6 m above the ground. [3]

Give your answers to three significant figures.

26. GDC A bakery bakes x cakes a day. Its daily profit, in dollars, is modelled by $P(x) = -0.35x^2 + 42x - 600$.

(a) Find the number of cakes the bakery should bake each day to maximise its profit, and state the maximum daily profit. [2]
(b) Find the numbers of cakes for which the bakery makes a profit. [3]
(c) Find the numbers of cakes for which the daily profit is more than $400. [2]

27. GDC Consider $f(x) = x^3 - 4x + 1$.

(a) Using your GDC, sketch the graph of f for $-3 \leq x \leq 3$, labelling all key features. [2]
(b) Find the zeros of f, giving your answers to 3 significant figures. [2]
(c) Find the coordinates of the local maximum and local minimum points. [2]
(d) Solve $f(x) = 2x - 1$, giving your answers to 3 significant figures. [2]

28. GDC The path of a javelin is modelled by $y = ax^2 + bx + c$, where x m is the horizontal distance from the point of release and y m is the height above level ground. The javelin is released at a height of 1.9 m, and it is 11.1 m high when $x = 20$ and 10.7 m high when $x = 40$.

(a) Write down three equations in a, b and c. [2]
(b) Use your GDC to find a, b and c. [2]
(c) Find the maximum height of the javelin. [2]
(d) Find the horizontal distance from the point of release to the point where the javelin lands. [2]
(e) Find the horizontal distance between the two points of the path at which the javelin is 8 m high. [2]

Give your answers to parts (c) to (e) to three significant figures.

29. A bridge has a parabolic arch. The arch is 40 m wide at road level and 10 m high at its centre. Take the road as the x-axis, with the centre of the arch at $x = 0$.

(a) Show that the arch is modelled by $y = 10 - \dfrac{x^2}{40}$. [4]
(b) Find the height of the arch 12 m from the centre. [2]
(c) A lorry 8 m wide and 9.5 m high drives along the centre of the road. Determine whether it passes under the arch. [4]
(d) Find the greatest width of a lorry 9 m high that can pass under the arch, giving your answer in the form $k\sqrt{10}$. [4]
(e) The model gives a negative value of y when $|x| > 20$. Explain what this means for the model. [2]

30. Consider the line $y = mx + 3$ and the parabola $y = x^2 + 5x + 7$.

(a) Show that at any point of intersection, $x^2 + (5 - m)x + 4 = 0$. [3]
(b) Find the values of m for which the line is a tangent to the parabola. [4]
(c) Find the values of m for which the line does not meet the parabola. [3]
(d) For the smaller value of m found in part (b), find the coordinates of the point of tangency. [3]
(e) State the values of m for which the line meets the parabola twice. [2]

31. HL Consider the system of equations

$$x + 2y - z = 1, \qquad 2x + 5y + z = 5, \qquad x + 4y + az = b,$$

where a and b are real constants.

(a) Write the system as an augmented matrix and row reduce it. [4]
(b) Find the value of a for which the system does not have a unique solution. [3]

(c) For that value of a, find the value of b for which the system is consistent. [2]
(d) For those values of a and b, find the general solution of the system. [4]
(e) Describe how the three planes are arranged in each of the three cases: a not equal to the value in part (b); a equal to it with b consistent; and a equal to it with b not consistent. [3]

Challenge Questions

32. Tangents from a point. A non-vertical line is a tangent to the parabola $y = x^2$ when it meets the parabola exactly once. End-of-Chapter Question 24(b) proves that $y = mx + c$ is a tangent exactly when $c = -\dfrac{m^2}{4}$; take that result as given here. This investigation asks how many tangents pass through a given point (h, k), and where the points with a perpendicular pair of tangents lie.

(a) Show that the line through (h, k) with gradient m is a tangent to $y = x^2$ if and only if

$$m^2 - 4hm + 4k = 0.$$

[3]

(b) Find the two tangents to $y = x^2$ that pass through the point $(3, 5)$, and state the point where each one touches the parabola. [4]
(c) Hence show that the number of tangents to $y = x^2$ through (h, k) is 2 when $k < h^2$, 1 when $k = h^2$ and 0 when $k > h^2$. Describe where the point lies relative to the parabola in each case, and explain each count from the picture. [5]
(d) Suppose that $k < h^2$, and let m_1 and m_2 be the gradients of the two tangents through (h, k). Write down $m_1 + m_2$ and $m_1 m_2$ in terms of h and k. Show that the two tangents are perpendicular if and only if $k = -\frac{1}{4}$. (The line $y = -\frac{1}{4}$ is called the *directrix* of the parabola.) [4]
(e) The acute angle θ between two lines with gradients m_1 and m_2, where $m_1 m_2 \neq -1$, satisfies $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$. Show that the points from which the two tangents meet at 45° satisfy $16h^2 = (1 + 4k)^2 + 16k$, and describe the curve these points form. Suggest how the investigation could continue, for example for other angles or for other parabolas. [4]

33. Three points fix a parabola. Two points fix a straight line. This investigation asks how many points are needed to fix a parabola $y = A + Bx + Cx^2$, and when a set of points fails to fix one. The unknowns are the coefficients A, B and C, and each point on the curve gives one linear equation in them.

(a) Find the parabola $y = A + Bx + Cx^2$ that passes through $(1, 1)$, $(2, 3)$ and $(3, 7)$. Write the three equations as an augmented matrix and row reduce it. [4]

(b) The third point is now replaced by $(k, 7)$, where k is a real constant. Show that row reduction leads to the last row

$$(k-1)(k-2)\,C = 8 - 2k,$$

and hence find C in terms of k whenever the system has a unique solution. [5]

(c) State the values of k for which the system does not have a unique solution. For each one, decide whether the system has no solution or infinitely many, and explain the result geometrically. [3]

(d) Find the value of k for which $C = 0$. Explain what the three points then have in common, and state the curve that passes through all three. [3]

(e) Conjecture how many points, with distinct x-coordinates, are needed to fix a polynomial of degree at most n. Explain your count by comparing the number of equations with the number of unknowns, and prove that two polynomials of degree at most n that agree at $n + 1$ distinct values of x are the same polynomial. *(Hint: a nonzero polynomial of degree at most n has at most n roots (Ch. 6))* [4]

(f) Without solving any equations, show that

$$p(x) = 1 \cdot \frac{(x-2)(x-3)}{(1-2)(1-3)} + 3 \cdot \frac{(x-1)(x-3)}{(2-1)(2-3)} + 7 \cdot \frac{(x-1)(x-2)}{(3-1)(3-2)}$$

passes through the three points of part (a), and that it simplifies to your answer to part (a). Explain how this construction generalises to $n + 1$ points, and suggest one question about fitting a polynomial through many points that could be investigated further. [4]