NUMBER & ALGEBRA

Chapter 2

Sequences & Series

2

SYLLABUS 1.2 1.3 1.4 1.8 1.9 1.10

A sheet of paper is about 0.1 mm thick. Fold it once and it is 0.2 mm. Each fold doubles the thickness, so 42 folds would give 0.1×2^{42} mm, which is about $440{,}000$ km. The Moon is about $384{,}000$ km away. Nobody can fold a sheet of paper 42 times, but the arithmetic is correct.

One further fold would reach the Moon and return. The reason lies in how doubling accumulates. Each fold doubles a thickness that has already been doubled, so after 42 folds the paper is not 42 times its original thickness but 2^{42} times it.

Now stack the sheets instead of folding them. Place one sheet at a time, and after 42 sheets the pile is 4.2 mm high. The paper is the same and the number of steps is the same, yet the answer is smaller by a factor of about a hundred billion. Folding multiplies at each step; stacking adds.

Those two operations produce the two kinds of list studied in this chapter. A list built by repeatedly adding a fixed amount is *arithmetic*. A list built by repeatedly multiplying by a fixed factor is *geometric*. The two arise in the same situations, so choosing the wrong one is costly: a salary that rises by a fixed amount each year is arithmetic, a salary that rises by a fixed percentage is geometric, and the two grow far apart over a career.

The same two patterns describe the value of a savings account and the growth of a population by a fixed percentage, and the counting methods at the end of the chapter give the expansion of $(a + b)^{12}$ without multiplying out twelve brackets. By the end of this chapter you will be able to find any term, and the sum of the first n terms, of an arithmetic or geometric sequence without writing out the terms before it; use sigma notation; find the sum of an infinite geometric series and explain why it can be finite; count arrangements and selections; and expand $(a + b)^n$ with the binomial theorem.

2.1 Arithmetic Sequences

A theatre has 20 seats in the front row, 23 in the second, and 26 in the third. Each row has 3 more seats than the one before it. To find how many seats are in row 50, counting up one row at a time would be slow. There is a clear pattern: each row adds 3 seats. A regular pattern like this can be written as a formula, and finding that formula is the point of this section.

A sequence where each term is obtained by adding a fixed constant to the previous one is called **arithmetic** (also called an **arithmetic progression**, or **AP**). That constant is the **common difference** d.

DEF An **arithmetic sequence** has the form $u_1,\ u_1+d,\ u_1+2d,\ \ldots$, where u_1 is the first term and d is the common difference.
For any two consecutive terms: $d = u_{n+1} - u_n$.

2.1.1 The general term

The n-th term is the first term plus $(n-1)$ steps of size d. Starting from u_1, you take one step to reach u_2, two steps to reach u_3, and so on.

KEY · IB

$$u_n = u_1 + (n-1)d$$

Worked through. To find u_{20} when the first term is 7 and the common difference is -3, take 19 steps of size -3 from $u_1 = 7$:

$$u_{20} = 7 + (20-1)(-3) = 7 - 57 = -50.$$

A negative common difference means the sequence is decreasing. The general term can be negative even when the first term is positive.

EXAMPLE 2.1

In an arithmetic sequence, $u_5 = 11$ and $u_9 = 27$. Find u_1 and d.
From u_5 to u_9 is exactly 4 steps:

$$4d = u_9 - u_5 = 27 - 11 = 16 \implies d = 4.$$

Then $u_1 = u_5 - 4d = 11 - 16 = -5$.

EXAMPLE 2.2

Find the number of terms in the arithmetic sequence $5, 9, 13, \dots, 101$.

Here $d = 4$. Set $u_n = 101$:

$$5 + (n-1)(4) = 101 \implies 4(n-1) = 96 \implies n = 25.$$

CAUTION The formula is $u_1 + (n-1)d$, not $u_1 + nd$. For the first term ($n = 1$), you add zero differences, not one.

YOUR TURN 2.1 Arithmetic Sequences

Finding a term

1. Find the term indicated.

(a) u_7 of $3, 7, 11, \dots$

(b) u_{15} of the AP with $u_1 = 12$, $d = -4$

(c) u_{10} of $-5, -1, 3, \dots$

(d) u_{25} of the AP with $u_1 = 100$, $d = -7$

Finding the first term or the common difference

2. Find the quantity indicated.

(a) d, given $u_1 = 5$ and $u_9 = 37$

(b) u_1, given $u_6 = 23$ and $d = 3$

(c) d, given $u_6 = 20$ and $u_{14} = -4$

(d) u_1, given $u_{12} = 50$ and $d = 4$

3. In an arithmetic sequence, $u_2 + u_5 = 31$ and $u_8 = 38$. Find u_1 and d.

Number of terms

4. Find the number of terms in each AP.

(a) $4, 11, 18, \dots, 144$

(b) $2, 9, 16, \dots, 100$

(c) $100, 96, 92, \dots, 4$

(d) $-7, -3, 1, \dots, 61$

Testing whether a number is a term

5. Show that the sequence $5, 11, 17, \ldots$ has a term equal to 101. State which term it is.

6. Determine whether each number is a term of the sequence $3, 10, 17, \ldots$. If it is, state which term it is.

(a) 150 (b) 200

Concept check

7. A student finds the 20th term of $5, 8, 11, \ldots$ as $u_{20} = 5 + 20 \times 3 = 65$. Identify the error, and find the correct value.

2.2 Arithmetic Series

When you need the total of the first n terms of an arithmetic sequence, adding them one at a time is always possible but rarely necessary. A pattern reduces the whole sum to one short calculation.

Consider adding all integers from 1 to 100. Write the sum forwards and backwards:

$$1 + 2 + 3 + \cdots + 99 + 100$$

$$100 + 99 + 98 + \cdots + 2 + 1$$

Every vertical pair sums to 101, and there are 100 such pairs. So twice the sum equals 100×101, giving $S_{100} = 5050$.

The same pairing works for any arithmetic series: the first and last term always sum to the same value, $u_1 + u_n$. There are n such pairs when you write the series twice, so $2S_n = n(u_1 + u_n)$, and therefore:

KEY · IB

$$S_n = \frac{n}{2}(u_1 + u_n) \qquad = \qquad \frac{n}{2}(2u_1 + (n-1)d)$$

The first form is faster when you know u_n. Substituting $u_n = u_1 + (n-1)d$ gives the second, which works when only u_1 and d are known.

Worked through. To find the sum of the first 30 terms of $2, 7, 12, \ldots$, note that $u_1 = 2$ and $d = 5$. The last term is not given, so use the second form:

$$S_{30} = \frac{30}{2}(2(2) + 29(5)) = 15(4 + 145) = 15 \times 149 = 2235.$$

When a formula for S_n is given, recover the terms from it: $u_1 = S_1$, and $u_n = S_n - S_{n-1}$ for $n \geq 2$, because the two sums differ only by their last term.

EXAMPLE 2.3

The sum of the first n terms of an arithmetic series is given by $S_n = 3n^2 + 5n$. Find u_1, d, and the general term u_n.

$u_1 = S_1 = 3 + 5 = 8$.

$u_2 = S_2 - S_1 = (12 + 10) - 8 = 14$, so $d = u_2 - u_1 = 6$.

For the general term: $u_n = S_n - S_{n-1} = 3n^2 + 5n - 3(n-1)^2 - 5(n-1)$.

Expanding: $u_n = 3n^2 + 5n - 3n^2 + 6n - 3 - 5n + 5 = 6n + 2$.

Check: $u_1 = 8$ and $u_2 = 14$. Consistent.

CAUTION S_n is the total of the first n terms; u_n is just the n-th term. Confusing the two is a common error. A question asking for a "term" wants u_n; a question asking for a "sum" wants S_n.

2.2.1 Application: simple interest

A standard financial application of an arithmetic sequence is **simple interest**: the same fixed amount of interest is added each year, so the balance increases by the same amount each year.

Write the opening balance as u_1 and the yearly interest as d. After n years, n payments of interest have been added, so the balance is

$$u_1 + nd.$$

Note the n here rather than the $n-1$ of the general term. The opening balance is the amount before any interest has been paid, so n years later there have been n payments, not $n-1$.

EXAMPLE 2.4

\$2000 is deposited in an account paying 3% simple interest per year. Find the balance after 15 years.

The interest is $3\% \times 2000 = \$60$ every year, so the balances form an arithmetic sequence with $d = 60$. After 15 years, 15 payments of interest have been added:

$$2000 + 15 \times 60 = \$2900.$$

2.2.2 When the data are not quite arithmetic

Real measurements rarely sit exactly on a sequence. A model is still useful if the steps are *close* to constant, and then the common difference has to be estimated rather than read off. A simple estimate is the average step: divide the total change by the number of steps.

KEY For data that are approximately arithmetic over n terms,

$$d \approx \frac{u_n - u_1}{n-1}.$$

This is the mean of the successive differences. Those differences telescope, so only the first and last readings survive: the estimate is quick, but it rests entirely on the two endpoints. Fitting a least-squares line, as in Ch. 10, uses every reading instead.

A model fitted to data is reliable only within the range the data covered. A prediction outside that range is an estimate, and a model of real growth will fail if it is carried far enough.

Worked through. A tree's height is measured at the end of each year: 1.8, 2.5, 3.1, 3.9, 4.6 metres. To test whether an arithmetic model is reasonable, look first at the successive differences: 0.7, 0.6, 0.8, 0.7. They are not equal, but they are close, so an arithmetic model is reasonable. Over four steps the height rose from 1.8 to 4.6, so

$$d \approx \frac{4.6 - 1.8}{5-1} = \frac{2.8}{4} = 0.7 \text{ metres per year.}$$

To predict the height after 8 years, take $u_1 = 1.8$ and $d = 0.7$; the eighth term is

$$u_8 = 1.8 + 7(0.7) = 6.7 \text{ metres.}$$

Treat that as an estimate, not a fact. Two things limit it. The model was built on five years and year 8 lies outside them, and no tree grows at a constant rate forever, so the model must fail eventually. A prediction is only as good as the range the data covered.

YOUR TURN 2.2 Arithmetic Series

Finding a sum

8. Find the sum indicated.

(a) S_{12} for the AP $2, 5, 8, \dots$

(b) S_{20} for the AP with $u_1 = 100$, $d = -5$

(c) S_{15} for the AP $7, 11, 15, \dots$

(d) S_{25} for the AP with $u_1 = -12$, $d = 5$

9. An AP has $u_1 = 15$, $u_n = -9$ and $n = 13$. Find S_n.

10. Find each sum.

(a) $6 + 10 + 14 + \cdots + 90$

(b) $3 + 8 + 13 + \cdots + 98$

(c) $50 + 46 + 42 + \cdots + 2$

(d) $1 + 1.5 + 2 + \cdots + 20$

11. Find the sum of all integers from 1 to 200 that are multiples of 6.

12. Find the sum $4 + 7 + 10 + \cdots + 100$.

Working backwards from a sum

13. Find n if $S_n = 234$ for the AP $3, 6, 9, \dots$

14. Given $S_n = 4n^2 - n$, find u_1, d and the general term u_n.

Simple interest

15. $3500 is deposited in an account paying 4% simple interest per year. Find the balance after 8 years.

16. An account pays simple interest. After 3 years the balance is $2360, and after 7 years it is $2840. Find the amount first deposited and the annual rate of interest.

Modelling with data

17. A plant's height is measured at the end of weeks 1 to 5, in centimetres: 4.2, 6.1, 7.8, 9.9, 11.8.

(a) Find the four successive differences, and explain why an arithmetic model is reasonable.

(b) Estimate the common difference from the first and last readings.

(c) Predict the height at the end of week 9, and give one reason to treat the prediction with caution.

Concept check

18. An arithmetic series has $S_n = n^2 + 3n$. Asked for the 10th term, a student writes $u_{10} = S_{10} = 130$. Identify the error, and find u_{10}.

2.3 Geometric Sequences

A geometric sequence multiplies by the same number at each step. Start at 1 and double: $1, 2, 4, 8, 16, \ldots$ Many real quantities change in this way. Bacteria double in every generation. A ball bounces to 60% of the height it fell from. Each extra filter that passes half the light reaching it halves the intensity again. In every case, each new value is the previous value multiplied by a fixed number.

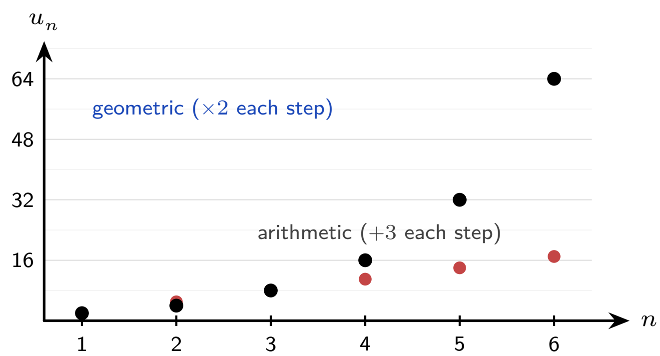

Figure 2.1 Adding a fixed amount grows in a straight line; multiplying by a fixed factor greater than 1 gives increases that themselves keep growing.

A sequence where each term is obtained by multiplying the previous term by a fixed constant is called **geometric** (also called a **geometric progression**, or **GP**). That constant is the **common ratio** r.

> DEF A **geometric sequence** has the form $u_1,\ u_1r,\ u_1r^2,\ \ldots$, where u_1 is the first term and r is the common ratio.
> For any two consecutive terms: $r = \dfrac{u_{n+1}}{u_n}$.

2.3.1 The general term

Starting from u_1, each step multiplies by r. After $(n-1)$ multiplications you reach u_n:

> KEY · IB
>
> $$u_n = u_1 \cdot r^{n-1}$$

Worked through. For the geometric sequence with $u_1 = 5$ and $r = 3$, the sixth term is

$$u_6 = 5 \cdot 3^5 = 5 \times 243 = 1215.$$

EXAMPLE 2.5

A geometric sequence has $u_3 = 28$ and $u_6 = 224$. Find u_1 and r.
From u_3 to u_6 is 3 multiplicative steps: $u_6 = u_3 \cdot r^3$, so

$$r^3 = \frac{224}{28} = 8 \implies r = 2.$$

Then $u_1 = u_3/r^2 = 28/4 = 7$.

EXAMPLE 2.6

Find the number of terms in the geometric sequence $2, 6, 18, \ldots, 4374$.
Here $r = 3$. Set $u_n = 4374$:

$$2 \cdot 3^{n-1} = 4374 \implies 3^{n-1} = 2187 = 3^7 \implies n = 8.$$

EXAMPLE 2.7

Given that k, 8 and $k + 12$ are consecutive terms of a geometric sequence, find the possible values of k.
Consecutive terms have a constant ratio, so the ratio across the first gap equals the ratio across the second:

$$\frac{8}{k} = \frac{k+12}{8} \implies 64 = k(k + 12) \implies k^2 + 12k - 64 = 0.$$

Factorising, $(k + 16)(k - 4) = 0$, so $k = -16$ or $k = 4$.
Both values give a genuine sequence. With $k = 4$ the terms are $4, 8, 16$ and $r = 2$; with $k = -16$ they are $-16, 8, -4$ and $r = -\frac{1}{2}$. Do not discard the negative solution: a negative ratio makes the terms alternate in sign, which is still a geometric sequence.

2.3.2 Recursively defined sequences

Every rule so far has been **explicit**: put n in and the n-th term comes out, with no need for any term before it. A sequence can instead be given by a **recurrence**: a starting value together with a rule that builds each term from the previous one. The sequence $u_1 = 4$, $u_{n+1} = 3u_n - 4$ is defined this way, and it runs $4, 8, 20, 56, \ldots$

A recurrence produces terms one at a time, so reaching u_{50} means computing the forty-nine terms below it. A closed form removes that work. For a rule of the shape $u_{n+1} = au_n + b$, the route to one is a shift: measure the terms from the value the rule leaves unchanged, and what remains is geometric.

Worked through. To find a closed form for the sequence $u_1 = 4$, $u_{n+1} = 3u_n - 4$, let

$v_n = u_n - 2$ and show that $\{v_n\}$ is geometric. Subtract 2 from both sides of the recurrence:

$$u_{n+1} - 2 = 3u_n - 6 = 3(u_n - 2), \qquad \text{that is} \qquad v_{n+1} = 3v_n.$$

So $\{v_n\}$ is geometric with $r = 3$, and $v_1 = u_1 - 2 = 2$. Therefore

$$v_n = 2 \cdot 3^{n-1}, \qquad u_n = v_n + 2 = 2 \cdot 3^{n-1} + 2.$$

Check the first three terms: $2 + 2 = 4$, $6 + 2 = 8$, $18 + 2 = 20$, matching 4, $3(4) - 4$, $3(8) - 4$. The substitution is not a guess. Solving $x = 3x - 4$ gives $x = 2$, the value the rule fixes, and $v_n = u_n - 2$ measures each term from it, which is exactly what cancels the -4.

2.3.3 Where geometric growth shows up

Three contexts show the pattern, and the algebra is the same in each.

- **Population growth.** A population rising by a fixed *percentage* each year is geometric with $r = 1 + \frac{p}{100}$. A 2% annual rise gives $r = 1.02$.
- **Salary change.** An annual rise of 3% gives $r = 1.03$; a cut of 3% gives $r = 0.97$. Note that a rise then an equal cut does not return you to the start, because $1.03 \times 0.97 = 0.9991$.
- **The spread of a disease.** If each infected person passes it to r others, the number of new cases in successive rounds is geometric. That r is what public health reporting calls the reproduction number, and whether it is above or below 1 decides whether the outbreak grows or dies out.

The last one shows why the ratio matters more than the starting value. If $r > 1$ the terms grow without limit, however small u_1 is; if $0 < r < 1$ they shrink towards zero, however large u_1 is. A large outbreak with $r = 0.9$ dies out, and a single case with $r = 1.1$ does not.

Two points of reading need care. A value n periods after the start has been multiplied n times, so it is $u_1 r^n$, not $u_n = u_1 r^{n-1}$; read the question for whether it counts terms or periods. When n counts whole periods, as in the first year a value is passed, round the solution of the equation up, whatever the decimal.

Worked through. A town of $40{,}000$ people grows by 2.5% each year. To find the population after 10 years, and the first year in which it exceeds $60{,}000$, take $u_1 = 40{,}000$ and $r = 1.025$. Ten years later is ten multiplications, so the value is $u_1 r^{10}$, not u_{10}:

$$40{,}000 \times 1.025^{10} \approx 51{,}203.$$

For the second part, solve $40{,}000 \times 1.025^n > 60{,}000$, that is $1.025^n > 1.5$. Taking logarithms (Ch. 1),

$$n > \frac{\ln 1.5}{\ln 1.025} \approx 16.4,$$

so the population first exceeds $60{,}000$ during year 17. Round *up* here whatever the decimal:

year 16 is not yet enough.

TIP To test whether a sequence is arithmetic, check that differences are constant. To test whether it is geometric, check that ratios are constant. A sequence usually passes at most one of these tests. The exception is a constant non-zero sequence such as $5, 5, 5, \ldots$, which is arithmetic with $d = 0$ and geometric with $r = 1$.

YOUR TURN 2.3 Geometric Sequences

Finding a term

19. Find the term indicated.

(a) u_6 of the GP $2, 6, 18, \ldots$

(b) u_7 of the GP with $u_1 = 96$, $r = \frac{1}{2}$

(c) u_5 of $\frac{1}{4}, \frac{1}{2}, 1, \ldots$

(d) u_8 of the GP with $u_1 = 3$, $r = -2$

Finding the first term or the common ratio

20. Find the quantity indicated.

(a) r, given $u_1 = 5$ and $u_4 = 40$

(b) u_1, given $u_4 = 24$ and $r = 2$

(c) r, given $u_1 = 81$, $u_5 = 1$ and $r > 0$

(d) u_1, given $u_3 = 45$ and $r = 3$

21. Given $u_2 = 10$ and $u_5 = 1250$, find u_1 and r.

22. The 2nd term of a GP is 12 and the 4th term is 48. Find the two possible pairs of values of u_1 and r.

Number of terms

23. Find the number of terms in each GP.

(a) $3, 6, 12, \ldots, 768$

(b) $5, 15, 45, \ldots, 3645$

(c) $5, -10, 20, \ldots, -2560$

(d) $128, 64, 32, \ldots, \frac{1}{2}$

Three consecutive terms

24. Given that k, 10 and $k - 15$ are consecutive terms of a geometric sequence, find the possible values of k and the common ratio in each case.

25. Given that x, $x + 6$ and $9x$ are consecutive terms of a geometric sequence, where $x \neq 0$, find the possible values of x. For each value, write down the three terms and the common ratio.

Geometric growth in context

26. GDC Geometric growth in three contexts.

(a) A population of $25{,}000$ grows by 1.8% each year. Find the population after 12 years.

(b) A salary of \$$48{,}000$ rises by 4% each year. Find the salary in the sixth year.

(c) In an outbreak each infected person infects 1.2 others, and round 1 has 50 new cases. Find the number of new cases in round 6, and state what would happen in the long run if that figure fell to 0.8.

27. GDC A town of $32{,}000$ people grows by 2.2% each year. Find the first year in which the population exceeds $45{,}000$.

Recurrences

28. A sequence is defined by $u_1 = 5$ and $u_{n+1} = 2u_n - 3$.

(a) Write down u_2, u_3 and u_4.

(b) Let $v_n = u_n - 3$. Show that $\{v_n\}$ is geometric.

(c) Hence find a closed form for u_n.

29. A sequence is defined by $u_1 = 10$ and $u_{n+1} = \frac{1}{2}u_n + 4$.

(a) Find the value of c for which $v_n = u_n - c$ is a geometric sequence.

(b) Find a closed form for u_n, and describe what happens to u_n as n increases.

Concept check

30. A town of 2000 people grows by 5% each year. A student models the population after n years as $P = 2000 \times 0.05^n$. Identify the error, and write down the correct model.

2.4 Geometric Series

Adding the terms of a geometric sequence needs a different method. In an arithmetic series, the first and last terms have the same total as the second and second-to-last terms, so pairing works. In a geometric series these pair totals in general differ, so pairing does not help. Instead there is an algebraic method: multiply the sum by r, line the two sums up, and subtract.

Let $S = u_1 + u_1r + u_1r^2 + \cdots + u_1r^{n-1}$. Multiply by r:

$$rS = u_1r + u_1r^2 + \cdots + u_1r^n.$$

Every term appears in both lines except u_1 in the first line and u_1r^n in the second. Subtract

rS from S:

$$S(1-r) = u_1 - u_1 r^n = u_1(1-r^n).$$

Divide both sides by $(1-r)$ to get the formula.

KEY · IB

$$S_n = \frac{u_1(1-r^n)}{1-r} \qquad = \qquad \frac{u_1(r^n-1)}{r-1}, \qquad r \neq 1$$

Both forms are equivalent. When $0 < r < 1$, use the first form: both $1 - r^n$ and $1 - r$ are then positive, so no negative signs appear. When $r > 1$ the second form is tidier, for the same reason. If r is negative, either form works, but r^n changes sign with n, so take care with the arithmetic.

Worked through. To find the sum of the first 10 terms of $5 + 10 + 20 + \cdots$, take $u_1 = 5$ and $r = 2$:

$$S_{10} = \frac{5(2^{10}-1)}{2-1} = 5 \times 1023 = 5115.$$

EXAMPLE 2.8

A geometric series has $u_1 = 80$ and $r = \frac{1}{2}$. Find the least value of n such that $S_n > 159$.

$$S_n = \frac{80\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = 160\left(1 - \frac{1}{2^n}\right).$$

Require $160(1 - \frac{1}{2^n}) > 159$, so $\frac{1}{2^n} < \frac{1}{160}$, giving $2^n > 160$.
Since $2^7 = 128 < 160$ and $2^8 = 256 > 160$, the least value is $n = 8$.

CAUTION The formula requires $r \neq 1$. If $r = 1$, every term equals u_1 and $S_n = nu_1$. Substituting $r = 1$ into the formula gives $0/0$, which is undefined.

Solving for n in geometric series often requires taking logarithms of both sides. See Ch. 1, §1.5.

YOUR TURN 2.4 Geometric Series

Finding a sum

31. Find the sum indicated.

(a) S_8 for the GP $1, 3, 9, \ldots$

(b) S_{10} for the GP with $u_1 = 6$, $r = \frac{1}{3}$

(c) S_6 for the GP $128, 64, 32, \ldots$

(d) S_6 for the GP with $u_1 = 1$, $r = -2$

32. Find each sum.

(a) $3 + 6 + 12 + \cdots + 1536$

(b) $2 + 6 + 18 + \cdots + 486$

(c) $1 - 2 + 4 - \cdots + 256$

(d) $96 + 48 + 24 + \cdots + 3$

Working backwards from a sum

33. The sum of the first three terms of a GP is 21 and the common ratio is 2. Find the first term.

34. Find n such that $S_n = 63$ for the GP $1, 2, 4, \ldots$

35. Given $S_3 = 7$ and $S_6 = 63$, find u_1 and r.

36. A geometric sequence has $u_1 = 5$ and $S_3 = 15$.

(a) Find the two possible values of r.

(b) For each value of r, find S_{10}.

The least number of terms

37. For the geometric series $2 + 6 + 18 + \cdots$, find the least value of n such that $S_n > 10\,000$.

38. GDC A geometric series has $u_1 = 60$ and $r = 0.8$. Find the least value of n such that $S_n > 290$.

Concept check

39. A student finds S_6 for the series $2 + 6 + 18 + \cdots$ as $\dfrac{2(1-3)^6}{1-3} = \dfrac{128}{-2} = -64$. Identify the error, and find S_6.

2.5 Sigma Notation

By now you have written several sums in the form $u_1 + u_2 + \cdots + u_n$. There is a compact notation for this that appears throughout mathematics, physics, and statistics, and it should be learned properly.

The Greek letter Σ (capital sigma) is used as shorthand for "sum". The expression

$$\sum_{k=1}^{n} u_k = u_1 + u_2 + u_3 + \cdots + u_n$$

is read "the sum of u_k from $k = 1$ to n." The variable k is the **index of summation**: it takes each integer value from the lower limit to the upper limit, the corresponding term is evaluated, and all terms are added.

DEF $\displaystyle\sum_{k=m}^{n} f(k) \;=\; f(m) + f(m+1) + f(m+2) + \cdots + f(n),$ where $m \le n$.
The index k is a **dummy variable**: the result depends only on f, m, and n, not on the name chosen for the index.

2.5.1 Properties of sigma notation

These three properties are just familiar rules of arithmetic, written in sigma form. They are what you use to simplify or split a sum.

KEY For a constant c and sequences $\{a_k\}$, $\{b_k\}$:

$\displaystyle\sum_{k=1}^{n} c = nc$	a constant summed n times
$\displaystyle\sum_{k=1}^{n} c\,a_k = c\sum_{k=1}^{n} a_k$	a constant factor comes outside the sum
$\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$	a sum splits over addition

When the lower limit is not 1, subtract the missing terms: for $m \ge 2$,

$$\sum_{k=m}^{n} f(k) = \sum_{k=1}^{n} f(k) - \sum_{k=1}^{m-1} f(k).$$

The sum has $n - m + 1$ terms.

Worked through. To write $3 + 7 + 11 + 15 + 19$ in sigma notation, notice that the terms form an arithmetic sequence with $u_1 = 3$, $d = 4$: the k-th term is $u_k = 4k - 1$. So

$$3 + 7 + 11 + 15 + 19 = \sum_{k=1}^{5}(4k - 1).$$

EXAMPLE 2.9

Evaluate $\displaystyle\sum_{k=1}^{6} 5 \cdot 2^{k-1}$.

This is a geometric series with $u_1 = 5$ and $r = 2$:

$$\sum_{k=1}^{6} 5 \cdot 2^{k-1} = S_6 = \frac{5(2^6 - 1)}{2 - 1} = 5 \times 63 = 315.$$

EXAMPLE 2.10

Evaluate $\sum_{k=3}^{10}(2k+1)$.

The lower limit is 3, not 1. Use the fact that the full sum minus the missing part gives the partial sum:

$$\sum_{k=3}^{10}(2k+1) = \sum_{k=1}^{10}(2k+1) - \sum_{k=1}^{2}(2k+1).$$

The first sum is an arithmetic series: $u_1 = 3$, $u_{10} = 21$, so $S_{10} = \frac{10}{2}(3 + 21) = 120$.

The second sum is just $3 + 5 = 8$.

Result: $120 - 8 = 112$.

CAUTION $\sum_{k=1}^{n}(a_k \cdot b_k) \neq \left(\sum a_k\right)\left(\sum b_k\right)$. You can split a sum over addition but not over multiplication. This is a common misapplication of sigma notation.

YOUR TURN 2.5 Sigma Notation

Writing a sum in sigma notation

40. Write each sum in sigma notation.

(a) $5 + 10 + 15 + 20 + 25$

(b) $4 + 8 + 16 + 32 + 64$

(c) $3 + 6 + 9 + \cdots + 30$

(d) $1 + 4 + 9 + 16 + 25$

Evaluating a sum

41. Evaluate each sum.

(a) $\sum_{k=1}^{6}(3k-2)$

(b) $\sum_{k=0}^{4} 81\left(\frac{1}{3}\right)^k$

(c) $\sum_{k=5}^{12}(3k-4)$

42. Consider the sum $\sum_{k=4}^{15}(5-2k)$.

(a) State the number of terms in the sum.

(b) Evaluate the sum.

(c) Rewrite the sum using the index $j = k-3$, so that j starts at 1, and check that it gives the same value.

Using the sigma properties

43. This question uses the properties of sigma notation.

(a) By treating it as an arithmetic series, show that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

(b) Hence find $\sum_{k=1}^{n} 5$ and $\sum_{k=1}^{n} 5k$.

44. Use the properties of sigma notation and the result of Question 43(a) to answer each part.

(a) Evaluate $\sum_{k=1}^{20}(3k+2)$.

(b) Evaluate $\sum_{k=11}^{30}(2k-1)$ by subtracting the terms that are missing.

(c) Evaluate $\sum_{k=1}^{3} k^2$ and $\left(\sum_{k=1}^{3} k\right)^2$. State what the two values show about sums of products.

Concept check

45. A student writes $\sum_{k=1}^{20}(3k+2) = 3\sum_{k=1}^{20} k + 2 = 3(210) + 2 = 632$. Identify the error, and find the correct value.

2.6 Infinite Geometric Series

Cut a pizza in half. Eat one half and keep the other. Cut what is left in half again, eat one piece, and keep the rest. If you could continue forever, how much pizza would you eat?

The pieces you eat are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, ... There are infinitely many of them, and every piece is larger than zero. Adding infinitely many positive numbers sounds like it must give an infinite total. But you began with one pizza, so you can never eat more than one pizza. So the total cannot be more than 1.

That is the surprise of this section: *an addition that never ends can still have a finite answer.*

Add the pieces one at a time and record the running total after each step. Each running total is a **partial sum**:

$$S_1 = 0.5, \quad S_2 = 0.75, \quad S_3 = 0.875, \quad S_4 = 0.9375, \quad \ldots$$

Two things happen at once. Every partial sum is larger than the one before it, and every partial sum is still smaller than 1. Look at the distance from 1: it is 0.5, then 0.25, then 0.125. The distance halves at every step, so it never becomes zero, but it becomes as small as you wish. This value that the partial sums approach is called the **limit**, and here the limit is 1.

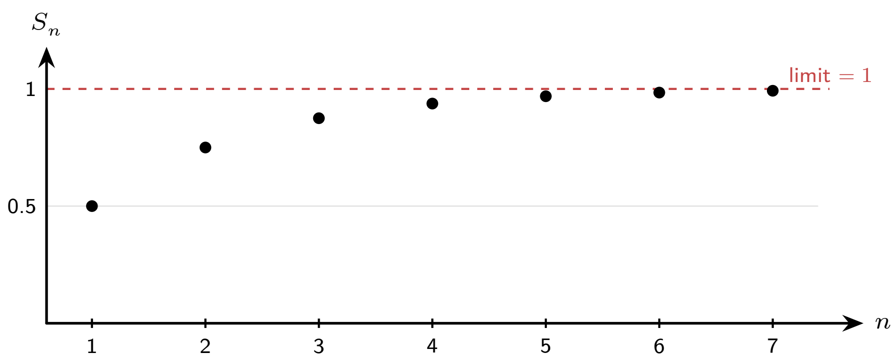

Figure 2.2 The partial sums of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ climb toward 1 but never pass it.

The pizza works because each piece is half the size of the piece before it. In general, $|r| < 1$ means that each term is smaller in size than the one before, by the fixed factor $|r|$, so the terms shrink toward zero.

Now change one number and the behaviour changes completely. Take $1 + 2 + 4 + 8 + \cdots$, where $r = 2$. Each term is larger than the last, the partial sums are $1, 3, 7, 15, \ldots$, and they pass every value you can name. There is no finite total.

A series whose partial sums approach a finite limit is said to **converge**. A series that does not converge is said to **diverge**: its partial sums may grow without bound, as here, or keep oscillating, as they do for $1 - 1 + 1 - 1 + \cdots$. A geometric series converges when $|r| < 1$ and diverges when $|r| \geq 1$.

To turn this into a formula, start from the finite sum and let n grow without bound. When $|r| < 1$, the power r^n shrinks toward 0:

$$S_n = \frac{u_1(1 - r^n)}{1 - r} \xrightarrow{n\to\infty} \frac{u_1(1-0)}{1-r} = \frac{u_1}{1-r}.$$

KEY · IB For a geometric series with $|r| < 1$:

$$S_\infty = \frac{u_1}{1-r}$$

A decimal that recurs from the decimal point is an infinite geometric series. The first term is the first copy of the repeating block, and $r = 10^{-m}$ for a block of m digits, so S_∞ gives the decimal as a fraction.

Worked through. To find the sum to infinity of $6 + 4 + \frac{8}{3} + \cdots$, first find the common ratio, $r = 4/6 = \frac{2}{3}$. Since $|r| < 1$ the series converges:

$$S_\infty = \frac{6}{1 - \frac{2}{3}} = \frac{6}{\frac{1}{3}} = 18.$$

EXAMPLE 2.11

Express $0.\overline{27} = 0.272727\ldots$ as an exact fraction.

Write as an infinite geometric series with $u_1 = \frac{27}{100}$ and $r = \frac{1}{100}$:

$$0.272727\ldots = \frac{27/100}{1 - 1/100} = \frac{27/100}{99/100} = \frac{27}{99} = \frac{3}{11}.$$

EXAMPLE 2.12

A geometric series has first term a and common ratio r. Given $S_\infty = 8$ and $u_2 = 2$, find a and r.

From S_∞: $\dfrac{a}{1-r} = 8$, so $a = 8(1 - r)$.

From $u_2 = ar = 2$: substitute a:

$$8(1 - r)r = 2 \implies 8r^2 - 8r + 2 = 0 \implies 4r^2 - 4r + 1 = 0 \implies (2r - 1)^2 = 0.$$

So $r = \frac{1}{2}$, and $a = 8\left(1 - \frac{1}{2}\right) = 4$.

CAUTION Always verify $|r| < 1$ before applying the infinite sum formula. If $|r| \geq 1$, no finite sum to infinity exists.

YOUR TURN 2.6 Infinite Geometric Series

Finding a sum to infinity

46. Find S_∞ for each series.

(a) $12 + 4 + \frac{4}{3} + \cdots$

(b) the GP with $u_1 = 10$, $r = \frac{3}{5}$

(c) $9 + 3 + 1 + \cdots$

(d) the GP with $u_1 = 8$, $r = -\frac{1}{4}$

47. A GP has $u_1 = 16$, $u_3 = 4$ and $r > 0$. Find S_∞.

Convergence

48. State whether each series converges, giving a reason. If it converges, find its sum to infinity.

(a) $6 + 9 + 13.5 + \cdots$

(b) $20 - 15 + 11.25 - \cdots$

Recurring decimals

49. Express each recurring decimal as an exact fraction in its lowest terms.

(a) $0.\overline{153} = 0.153153\ldots$

(b) $2.1\overline{6} = 2.1666\ldots$

Finding the first term or the common ratio

50. Find the quantity indicated.

(a) u_1, given $S_\infty = 20$ and $r = \frac{1}{5}$

(b) r, given $u_1 = 6$ and $S_\infty = 9$

(c) u_1, given $S_\infty = 12$ and $r = \frac{1}{4}$

(d) r, given $u_1 = 15$ and $S_\infty = 12$

51. A geometric series has $S_\infty = 27$ and $u_2 = 6$. Find the two possible pairs of values of u_1 and r.

52. A geometric series has $S_\infty = 16$, and the sum of its first two terms is 12. Find the possible values of r and the corresponding values of u_1.

Concept check

53. A student finds the sum to infinity of $1 + \frac{3}{2} + \frac{9}{4} + \cdots$ as $\dfrac{1}{1 - \frac{3}{2}} = -2$. Explain why this answer is wrong.

2.7 Financial Applications

A bank pays you 4% per year. In the first year you earn interest on your deposit. In the second year you earn interest on the deposit *and* on the interest from the first year. The amount added is different every year, so the balances do not form an arithmetic sequence. They form a geometric one: each year the balance is multiplied by 1.04.

This is **compound interest**, and it is why money left alone for a long time grows far more than it would under simple interest. The balance after any number of years is simply a term of a geometric sequence.

Interest is often added more than once a year. If a present value PV earns interest at a nominal annual rate of $r\%$, compounded k times per year, then after n years:

KEY · IB

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$$

The quantity $\left(1 + \frac{r}{100k}\right)$ is the common ratio of the balances (here r is the interest rate, not the ratio). The total number of compounding periods is kn.

Worked through. Suppose \$5 000 is invested at 4% per year, compounded monthly. After 3 years, with $PV = 5000$, $r = 4$, $k = 12$ and $n = 3$, its value is

$$FV = 5000 \times \left(1 + \frac{4}{1200}\right)^{36} \approx \$5636.36.$$

EXAMPLE 2.13

How many complete years does it take for an investment to double at 6% per year, compounded annually?

Set $FV = 2 \cdot PV$:

$$(1.06)^n = 2 \implies n = \frac{\ln 2}{\ln 1.06} \approx 11.9.$$

After 12 complete years the investment has more than doubled.

TIP The Rule of 72: dividing 72 by the annual interest rate gives an approximate doubling time in years. At 6%, that is $72/6 = 12$ years, matching the exact calculation above.

2.7.1 Regular deposits

A single lump sum gives one term of a geometric sequence. A savings plan is different: money goes in every year, and each payment then compounds for however long it has left in the account. The balance is a total of geometric terms, so it is a geometric *series*.

Write the payment as P and the annual rate as a decimal i. With payments made at the **end** of each year for n years, the final payment earns no interest at all, the one before it earns interest for one year, and the first earns interest for $n-1$ years. Reading the total from the last payment backwards gives a GP with first term P and common ratio $1+i$.

There is no new formula to learn: the sum is the geometric series of §2.4 with $u_1 = P$ and $r = 1+i$.

Payments made at the **start** of each year each spend one extra year in the account. Rather than remembering a correction factor, change the first term: the last payment now earns a year's interest, so $u_1 = P(1+i)$ and everything else is unchanged.

Worked through. Suppose \$800 is paid into an account at the end of each year for 12 years, at 5% per year compounded annually. To find the value immediately after the twelfth payment, start from the last payment. The payment made at the end of year 12 has earned nothing, the one at the end of year 11 has earned interest for a single year, and the first payment has earned interest for 11 years. Totalling from the last payment backwards:

$$800 + 800(1.05) + 800(1.05)^2 + \cdots + 800(1.05)^{11}.$$

This is a geometric series with $u_1 = 800$, $r = 1.05$ and $n = 12$ terms:

$$S_{12} = \frac{800(1.05^{12} - 1)}{1.05 - 1} \approx \$12\,733.70.$$

Had the same payments been made at the *start* of each year, every one of them would earn a further year of interest, so $u_1 = 800(1.05)$ and the same sum gives

$$S_{12} = \frac{800(1.05)(1.05^{12} - 1)}{0.05} \approx \$13\,370.39.$$

Settle which end of the year the payments land on before writing down u_1; the two answers differ by a full year's interest on the whole plan.

2.7.2 Depreciation

Some things lose value instead of gaining it. A car, a laptop, or a machine may lose a fixed *percentage* of its value each year. This is called **depreciation**.

No new formula is needed. Use the same compound-interest formula with a **negative** rate r. A loss of 12% per year means $r = -12$, so the common ratio becomes

$$1 + \frac{r}{100} = 1 - \frac{12}{100} = 0.88,$$

a number less than 1. The value is still multiplied by the same number each year, so the sequence is still geometric; because that number is less than 1, the values shrink.

2.7.3 Inflation and real value

There is a second geometric process working against your savings. **Inflation** is the yearly rise in prices. Interest increases the number in your account, while inflation reduces what that number can buy. If your account grows by 2% in a year and prices rise by 3% in the same year, you hold more money but you can buy less with it.

The **real value** of an investment is its future value adjusted for the price rises over the same period. It measures the gain in purchasing power rather than the gain in the number.

Prices climb geometrically too, so the correction is a second geometric factor, this time dividing rather than multiplying.

KEY With interest producing a future value FV over n years, and prices rising at $i\%$ per year:

$$\text{real value} = \frac{FV}{\left(1+\frac{i}{100}\right)^n}.$$

Divide by the inflation factor, because the same goods cost $\left(1+\frac{i}{100}\right)^n$ *times as much after* n *years.*

Worked through. Suppose \$10 000 is invested for 5 years at 4.5% per year, compounded annually, while prices rise at 2.5% per year. To find the future value and the real value of the investment, first note that the balance grows by a factor of 1.045 each year:

$$FV = 10\,000 \times (1.045)^5 \approx \$12\,461.82.$$

Now adjust for five years of price rises:

$$\text{real value} = \frac{12\,461.82}{(1.025)^5} \approx \$11\,014.43.$$

The account shows a gain of about \$2460, but in purchasing power the investor is only about \$1014 better off. An interest rate means little until you compare it with inflation.

TIP Your GDC's built-in Finance (TVM) solver handles compound interest, depreciation, and unknown rates or times directly: enter the known values of N, $I\%$, PV, FV, and the compounding frequency, and solve for the missing one. In examinations, state the values you entered.

Solving for n in financial problems requires exponential equations. See Ch. 1, §1.5.

YOUR TURN 2.7 Financial Applications

Compound interest

54. GDC Find the future value of each investment.

(a) \$3 000 at 4% p.a., compounded annually, for 5 years

(b) \$8 000 at 6% p.a., compounded monthly, for 3 years

(c) \$500 at 3.6% p.a., compounded monthly, for 10 years

(d) \$2 000 at 5% p.a., compounded quarterly, for 4 years

55. GDC Find the present value needed to accumulate \$15 000 in 8 years at 5% p.a., compounded quarterly.

Unknown rate or time

56. GDC Find the annual interest rate if \$5 000 grows to \$5 828.84 in 3 years, compounded annually.

57. GDC How many complete years does it take an investment to triple at 7% p.a., compounded monthly?

Depreciation

58. GDC A car valued at \$25 000 depreciates at 15% per year. Find its value after 4 years.

59. GDC A machine bought for \$80 000 is worth \$41 000 after 5 years. It loses the same percentage of its value each year. Find that percentage.

60. GDC A car bought for \$28 000 loses 12% of its value each year.

(a) Find its value after 5 years.

(b) Explain why this model never gives the car a value of zero.

Regular deposits

61. GDC \$1200 is paid into an account every year for 10 years. The account pays 4.5% p.a., compounded annually.

(a) The payments are made at the end of each year. Find the value immediately after the tenth payment.

(b) The payments are made at the start of each year instead. Find the value at the end of the tenth year.

62. GDC A saver wants to have \$50 000 after 15 years. She pays equal amounts into an account at the end of each year. The account pays 5% p.a., compounded annually. Find the size of each payment.

Inflation and real value

63. GDC $6000 is invested for 8 years at 3.8% p.a., compounded annually. Prices rise at 2.1% per year.

(a) Find the future value of the investment.

(b) Find its real value.

64. GDC Prices rise at 2.8% per year. A sum of money is kept without earning any interest. Find the number of complete years after which it has lost at least a quarter of its purchasing power.

Concept check

65. GDC A student finds the value of $5000 after 3 years at 6% p.a., compounded monthly, as 5000×1.06^{36}. Identify the error, and find the correct value.

2.8 Counting Principles HL

A four-digit PIN is built from the digits 0 to 9, and any digit may repeat. There are 10 choices for the first digit, 10 for the second, 10 for the third and 10 for the fourth, so the number of PINs is

$$10 \times 10 \times 10 \times 10 = 10\,000.$$

Writing the list out would take ten thousand lines. The total comes from the way the choices are made, and two ordinary words decide what to do with the counts.

And multiplies. When a result needs one choice *and* then another, the counts multiply. A menu offering 3 starters and 2 mains gives $3 \times 2 = 6$ meals.

A tree shows why. Each starter has the same number of branches below it, so the total is one count multiplied by the other rather than added.

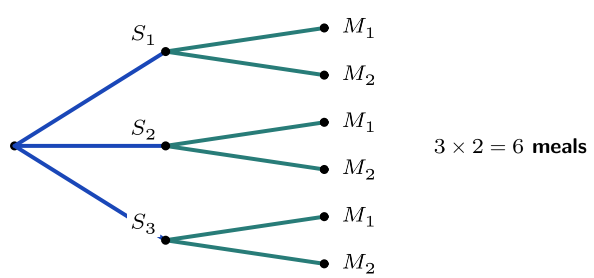

Figure 2.3 Three starters, two mains for each. Every path is a starter *and* a main, and there are $3 \times 2 = 6$ of them: that is what "and multiplies" looks like.

Add a third course and the same reasoning applies again: with 4 desserts the count becomes $3 \times 2 \times 4 = 24$. Every extra choice multiplies.

Or adds. Not every count is a chain of choices. Sometimes the answer is one thing *or* another, and the two cannot both happen: a password of length 3 *or* length 4, a team with three women *or* four. Count each possibility separately and add.

Those two words decide how the counts combine. Everything else in this section is about counting a single choice correctly once you know which word applies.

DEF HL **And multiplies** (the **multiplication principle**). If a result needs a first choice *and* a second, made in m and n ways, there are $m \times n$ results. Any number of choices chains the same way.
The condition: n must be the same number whichever way the first choice went. It may differ from m (10 people, then 9), but it may not depend on which one you picked.

Or adds (the **addition principle**). If the result is one case *or* another, add the counts.
The condition: the cases must not overlap. Nothing may be counted twice.

CAUTION Everyday English uses *or* inclusively: "students who take French or Spanish" includes those taking both. That "or" does not simply add, because the students taking both would be counted twice. In this section *or* always means the exclusive one: this case or that case, never both. The overlapping kind is handled in Ch. 11, §11.3, where $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ subtracts the double count.

Arranging n different objects in a row is one long chain of *and*s: n choices for the first position, $n-1$ for the second, and so on down to 1. The product is n **factorial**.

KEY HL

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1, \qquad 0! = 1$$

The value $0! = 1$ is set by definition, not derived. There is one way to arrange no objects at all, and fixing $0! = 1$ keeps every formula below correct at its end points.

2.8.1 Permutations and combinations

There are two standard questions about selecting r objects from n different ones. If the *order* of selection matters, the count is a **permutation**. If only the final collection matters, it is a **combination**.

For a permutation, stop the factorial early. There are n choices for the first object, $n-1$ for the second, and so on for r factors in total. Multiplying and dividing by the remaining $(n-r)$ factors turns this product into a ratio of factorials:

$${}^{n}P_{r} = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

The combination count follows from the permutation count. Choose 3 letters from A, B, C, D, E. The selection $\{A, B, C\}$ can be written in $3! = 6$ orders: ABC, ACB, BAC, BCA, CAB, CBA. A permutation count records six results here; a combination count records one. Every combination is therefore counted $r!$ times among the permutations, so dividing by $r!$ removes the duplicates:

$$^{n}C_{r} = \frac{^{n}P_{r}}{r!}.$$

KEY · IB HL Selecting r from n distinct objects:

$$^{n}P_{r} = \frac{n!}{(n-r)!} \quad \text{(ordered)}, \qquad ^{n}C_{r} = \frac{n!}{r!\,(n-r)!} \quad \text{(unordered)}$$

The symbol $^{n}C_{r}$ is read "n choose r". Other books write the same number as $\binom{n}{r}$, so both forms should be recognised; the formula booklet uses $^{n}C_{r}$, and a GDC labels the two counts nPr and nCr.

The question	The count	Example
Arrange all n objects	$n!$	6 books in a row
Arrange r of the n, order matters	$^{n}P_{r}$	3 medals for 8 runners
Choose r of the n, order does not matter	$^{n}C_{r}$	a committee of 3 from 12

Three steps are enough for almost every counting question in this course.

1. Read the question for *and* or *or*. *And* multiplies, *or* adds. Check the condition each one carries.
2. Within a single choice, ask whether the order matters. If it does, use $^{n}P_{r}$. If it does not, use $^{n}C_{r}$.
3. If the restriction is awkward to count directly, count everything and subtract what breaks it.

Two restrictions appear often enough to name. When certain objects must stay together, treat the whole group as one object, arrange the objects, then arrange the group internally and multiply the two counts. When a question asks for *at least one* of something, step 3 is usually shorter: subtract the arrangements that have none from the total.

Worked through. When eight runners compete for gold, silver and bronze, the three medals are distinguishable, so order matters. The number of ways to award them is

$$^{8}P_{3} = 8 \times 7 \times 6 = 336.$$

EXAMPLE 2.14

A committee of 5 is chosen from 6 boys and 5 girls. How many committees contain exactly 2 girls?

The committee needs 2 girls *and* 3 boys, so the two counts multiply. A committee has no internal order, so each count is a combination:

$$^5C_2 \times {}^6C_3 = 10 \times 20 = 200.$$

EXAMPLE 2.15

Five people sit in a row. In how many ways can they sit if two of them refuse to sit next to each other?

Count the seatings that place the two together, then subtract. Treat the pair as one object: four objects give $4! = 24$ arrangements, and the pair can sit in $2! = 2$ orders inside their block, so

$$24 \times 2 = 48$$

seatings place them together. All five can be seated in $5! = 120$ ways, so

$$120 - 48 = 72$$

seatings keep them apart. Counting the forbidden cases and subtracting is shorter here than building the allowed ones directly.

CAUTION Ask one question before every counting problem: *does order matter?* Medals, codes and rankings: yes, so use ^nP_r. Committees, hands of cards and subsets: no, so use ^nC_r.

TIP A GDC computes both, under the labels nPr and nCr. Two kinds of counting are not required in this course: arrangements in which some of the objects are identical, and arrangements in a circle.

YOUR TURN 2.8 Counting Principles HL

Arrangements and selections

66. Find each count.

(a) arrangements of 6 different books in a row

(b) 4-digit codes using the digits 1 to 9, no digit repeated

(c) committees of 3 chosen from 12 people

(d) groups of 2 men and 2 women chosen from 7 men and 6 women

Restrictions

67. Seven different people stand in a row. Find the number of arrangements in which

(a) three particular people stand next to each other

(b) two particular people stand at the two ends of the row

68. A 4-digit code uses the digits 1 to 9 with no digit repeated. Find the number of such codes that end in an even digit.

69. A committee of 3 is chosen from 12 people, one of whom is Ana. Find the number of committees that

(a) include Ana

(b) do not include Ana

Adding separate cases

70. Each of these is an *or*: split the count into parts that cannot both happen, then add.

(a) A group of 4 is chosen from 7 men and 6 women. Find the number of groups containing at least 3 women.

(b) A password is 3 or 4 letters long, chosen from the 26 lowercase letters with no letter repeated. Find the number of possible passwords.

At least one

71. A committee of 4 is chosen from 5 men and 7 women. By subtracting the committees with no men from the total, find the number of committees with at least one man.

72. A 4-digit code uses the digits 0 to 9, and digits may be repeated. Find the number of codes that contain at least one 7.

Solving for n

73. Solve each equation for the positive integer n.

(a) ${}^{n}C_2 = 45$

(b) ${}^{n}P_2 = 56$

Concept check

74. A student says the number of committees of 3 that can be chosen from 10 people is ${}^{10}P_3 = 720$. Explain the error, and find the correct number.

75. A committee of 4 is chosen from 5 men and 7 women. To count committees with at least one man, a student chooses one man (5 ways) and then any 3 of the remaining 11 people, giving $5 \times {}^{11}C_3 = 825$.

(a) Explain why this overcounts.

(b) Find the correct number.

2.9 The Binomial Theorem

Expanding $(a+b)^2$ is quick. Expanding $(a+b)^3$ takes longer. Expanding $(a+b)^{12}$ by multiplying out twelve brackets is not realistic. Fortunately the answer follows a pattern, so no bracket multiplication is needed at all.

Look at the small cases:

$$(a+b)^1 = a+b, \qquad (a+b)^2 = a^2+2ab+b^2, \qquad (a+b)^3 = a^3+3a^2b+3ab^2+b^3.$$

Two features are regular. The power of a falls from n to 0 while the power of b rises from 0 to n, and in every term the two powers add to n. The coefficients form a row of **Pascal's triangle**, in which each row begins and ends with 1 and every other entry is the sum of the two entries above it.

Number the rows from zero. The top row is row 0, the next is row 1, and so on, so that **row n holds the coefficients of** $(a+b)^n$. The top row is not decoration: $(a+b)^0 = 1$, which is the single entry 1. Number the entries along a row from zero as well, so that the entry in position r of row n multiplies $a^{n-r}b^r$.

$$\begin{array}{lcccccccccccl}
n=0 & & & & & & 1 & & & & & & (a+b)^0 \\
n=1 & & & & & 1 & & 1 & & & & & a+b \\
n=2 & & & & 1 & & 2 & & 1 & & & & a^2+2ab+b^2 \\
n=3 & & & 1 & & 3 & & 3 & & 1 & & & a^3+3a^2b+3ab^2+b^3 \\
n=4 & & 1 & & 4 & & 6 & & 4 & & 1 & & r=0,1,2,3,4 \text{ along the row} \\
n=5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 &
\end{array}$$

With both counts starting at zero, the label matches the algebra exactly: to expand $(a+b)^4$, read row 4, and its entry in position r is the coefficient of $a^{4-r}b^r$. Row n has $n+1$ entries, which is the number of terms in the expansion.

2.9.1 Why the entries are the combinations

Multiplying out $(a+b)^n$ means going through the n brackets and choosing either a or b from each one, then adding every product so formed. A product ends up as $a^{n-r}b^r$ exactly when b was chosen from r of the brackets and a from the other $n-r$. So the coefficient of $a^{n-r}b^r$ counts one thing: **how many ways there are to choose which r of the n brackets supply the b**. Choosing r objects from n, order not mattering, is what ${}^nC_r$ counts §2.8, so the coefficient *is* ${}^nC_r$.

The case $n=3$ is small enough to check by hand. Writing the three brackets as $(a+b)(a+b)(a+b)$, the term a^2b appears when exactly one bracket gives a b, and there are three ways for that to happen: the first bracket, the second, or the third. So the coefficient is 3, and ${}^3C_1 = 3$. Likewise ab^2 needs two brackets to give a b, which can be done in ${}^3C_2 = 3$ ways, while a^3 and b^3 each arise in only one way: ${}^3C_0 = {}^3C_3 = 1$.

That also explains the addition rule that builds the triangle. Pick one bracket and ask whether it supplies a b. If it does, the other $r-1$ copies of b come from the remaining $n-1$ brackets, in ${}^{n-1}C_{r-1}$ ways; if it does not, all r come from those $n-1$ brackets, in ${}^{n-1}C_r$ ways. The two cases cover every selection once, so

$${}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r,$$

which is exactly "each entry is the sum of the two entries above it". Since both descriptions start each row with 1 and end it with 1 (${}^{n}C_0 = {}^{n}C_n = 1$) and then grow by the same rule, the triangle and the combinations agree row for row.

KEY · IB For $n \in \mathbb{N}$:

$$(a+b)^n = a^n + {}^{n}C_1\, a^{n-1}b + \cdots + {}^{n}C_r\, a^{n-r}b^r + \cdots + b^n,$$

$$\text{where} \quad {}^{n}C_r = \frac{n!}{r!\,(n-r)!}.$$

Pascal's triangle is fastest for small n, provided you count rows from zero so that row n is the one you want. When n is large, or when a question asks for one coefficient only, use the single term ${}^{n}C_r\, a^{n-r}b^r$ shown in the middle of the expansion above and choose the r you need.

TIP The booklet shows that single term inside the expansion without naming it. The **general term** is

$$T_{r+1} = {}^{n}C_r\, a^{n-r}b^r,$$

so $r=0$ gives the first term, $r=1$ the second, and r gives term number $r+1$. The subscript is $r+1$ because r counts how many factors of b the term contains, and the first term contains none. In an examination, write down this term, find the r that produces the power you want, then substitute.

A question asking for the **constant term**, or for the term *independent of* x, is asking for the one in which **the power of** x **is** 0, since $x^0 = 1$ leaves a number. Simplify the general term until x appears once, as a single power, then set that power equal to 0 and solve for r.

Worked through. To expand $(x+2)^4$, take $n=4$ and read row 4 of the triangle, the fifth row down: $1, 4, 6, 4, 1$, with the powers of 2 increasing as the powers of x decrease:

$$(x+2)^4 = x^4 + 4x^3(2) + 6x^2(4) + 4x(8) + 16 = x^4 + 8x^3 + 24x^2 + 32x + 16.$$

EXAMPLE 2.16

Find the coefficient of x^3 in $(2x-3)^5$.

The general term is ${}^5C_r\,(2x)^{5-r}(-3)^r$. For x^3, take $5-r=3$, so $r=2$:

$${}^5C_2\,(2x)^3(-3)^2 = 10\times 8x^3\times 9 = 720x^3.$$

The coefficient is 720. Keep the sign inside the bracket with $b=-3$: the $(-3)^r$ produces the alternating signs automatically.

EXAMPLE 2.17

Find the term independent of x in $\left(x^2+\dfrac{2}{x}\right)^6$.

Track the power of x in the general term:

$$T_{r+1} = {}^6C_r\,(x^2)^{6-r}\left(\frac{2}{x}\right)^r = {}^6C_r\,2^r\,x^{12-3r}.$$

Independent of x means $12-3r=0$, so $r=4$:

$${}^6C_4\,2^4 = 15\times 16 = 240.$$

TIP ${}^nC_r$ can also be read from a GDC table. To solve ${}^6C_r=20$, generate ${}^6C_r$ for $r=0,\dots,6$ and read off $r=3$. Symmetry, ${}^nC_r={}^nC_{n-r}$, halves the search.

YOUR TURN 2.9 The Binomial Theorem

Expanding

76. Expand or write down the terms requested.

(a) Expand $(2+y)^5$, using row 5 of Pascal's triangle.

(b) Write down the first three terms, in ascending powers of x, of $(1+2x)^7$.

(c) Expand $(2x-1)^4$.

(d) Write down the first three terms, in ascending powers of x, of $(1-3x)^6$.

A single coefficient

77. Find the coefficient requested.

(a) x^4 in $\left(3-\frac{x}{3}\right)^6$

(b) x^3 in $(2+x)^6$

(c) x^4 in $(1-2x)^7$

(d) x^5 in $(x+3)^8$

78. Find the middle term in the expansion of $(x-2y)^6$.

Term numbers and symmetry

79. Answer each part without expanding in full.

(a) Write down the 5th term of the expansion of $(1+x)^{10}$.

(b) Given that ${}^{12}C_5 = 792$, write down ${}^{12}C_7$, giving a reason.

(c) In the expansion of $(1+x)^n$, the coefficients of x^3 and x^7 are equal. Find n.

Terms with a negative power of x

80. Find the term requested.

(a) the term independent of x in $\left(x+\dfrac{1}{x^2}\right)^9$

(b) the term independent of x in $\left(2x-\dfrac{1}{x}\right)^6$

(c) the coefficient of x^3 in $\left(x-\dfrac{2}{x}\right)^7$

(d) the term independent of x in $\left(3x^2-\dfrac{1}{x}\right)^6$

Concept check

81. A student says the coefficient of x^3 in $(1-2x)^5$ is ${}^5C_3 \times 2^3 = 80$. Identify the error, and find the correct coefficient.

2.10 The Binomial Series HL

So far n has been a positive whole number, because $(a+b)^n$ meant n brackets multiplied together. But expressions such as $\sqrt{1+x} = (1+x)^{1/2}$ and $\dfrac{1}{1+x} = (1+x)^{-1}$ also have exponents. Can they be expanded too?

They can, and the same coefficient pattern still works for *any* rational n. Two things change. The expansion never ends, and it is only valid when x is small enough.

Where the extension comes from. Write the coefficient of §2.9 without factorials. In

$${}^nC_r = \frac{n!}{r!\,(n-r)!},$$

the $(n-r)!$ cancels all but the first r factors of $n!$, leaving

$${}^nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!},$$

which is r factors counting down from n, divided by $r!$. Now look at what each side needs. The factorial $n!$ only makes sense for a non-negative whole number, but the product of r descending factors makes sense for *any* n: with $n=\frac{1}{2}$ and $r=2$ it is $\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} = -\frac{1}{8}$, and with $n=-1$ and $r=3$ it is $\frac{(-1)(-2)(-3)}{3!}=-1$. Keeping the coefficients in this form, and letting n be any rational number, is the whole of the extension. For a whole n nothing has changed: at $n=3$, $r=2$ the formula gives $\frac{3\times 2}{2!}=3={}^3C_2$.

What is lost is the counting. For a whole n the coefficient counts brackets; for $n=\frac{1}{2}$ it counts nothing, and the equality with the function has to be earned another way, which is where the condition on x comes in.

KEY · IB HL For $n \in \mathbb{Q}$:

$$(a+b)^n = a^n\left(1+n\frac{b}{a}+\frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2+\cdots\right)$$

The booklet omits the validity condition. When n is not a non-negative integer the series never ends, so it is valid only when $\left|\frac{b}{a}\right|<1$: factor out the term of larger size to make this hold. When $n\in\mathbb{N}$ the series stops and needs no condition.

In practice you first factor the expression into the form $(1+\text{something})^n$, which turns the booklet formula into the version you actually compute with:

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots, \qquad |x|<1 \text{ when } n\notin\mathbb{N}.$$

Why does the expansion stop for whole numbers but not otherwise? Look at those descending factors $n(n-1)(n-2)\cdots$ again. If $n=3$, the fourth factor is $n-3=0$, so every later coefficient is zero and the series ends: that is the finite theorem of §2.9. If $n=\frac{1}{2}$ or $n=-1$, no factor is ever zero, so the terms continue forever.

The condition $|x|<1$ should look familiar. Put $n=-1$:

$$(1+x)^{-1} = 1-x+x^2-x^3+\cdots$$

This is a geometric series with first term 1 and common ratio $-x$, and §2.6 says it converges to $\frac{1}{1+x}$ exactly when $|-x|<1$. So in this case the condition is not an extra assumption: it is the convergence condition already proved, and the same bound turns out to hold for every rational n. The two infinite series of this chapter are the same idea seen from two directions.

Worked through. To expand $(1-3x)^{1/3}$ up to and including the term in x^2, apply the series with $n=\frac{1}{3}$ and x replaced by $-3x$:

$$(1-3x)^{1/3} = 1+\tfrac{1}{3}(-3x)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}(-3x)^2+\cdots = 1-x-x^2-\cdots$$

Validity requires $|-3x|<1$, that is $|x|<\frac{1}{3}$.

When the bracket does not already start with 1, factor out the larger term first, as the booklet formula does. The expansion is then valid for $\left|\frac{b}{a}\right|<1$.

EXAMPLE 2.18

Expand $\dfrac{1}{2+x}$ in ascending powers of x up to x^2, stating the validity.
Factor out the 2:

$$(2+x)^{-1} = \frac{1}{2}\left(1+\frac{x}{2}\right)^{-1} = \frac{1}{2}\left(1-\frac{x}{2}+\frac{x^2}{4}-\cdots\right) = \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\cdots$$

valid for $\left|\frac{x}{2}\right| < 1$, that is $|x| < 2$.

A truncated expansion gives an approximation. Choose x inside the interval of validity so that the bracket becomes the value you want, or a value it can be found from. The closer x is to 0, the faster the later terms shrink and the better the approximation.

EXAMPLE 2.19

By expanding $(1-x)^{1/2}$ up to x^2 and choosing a suitable value of x, find an approximation to $\sqrt{2}$.
The expansion is

$$(1-x)^{1/2} = 1-\frac{x}{2}-\frac{x^2}{8}-\cdots$$

Now pick x so that $1-x$ connects to 2. Take $x = 0.02$: then $1-x = 0.98 = \frac{49}{50}$, so

$$\sqrt{0.98} = \frac{7}{\sqrt{50}} = \frac{7\sqrt{2}}{10} \implies \sqrt{2} = \frac{10}{7}\sqrt{0.98}.$$

The series gives $\sqrt{0.98} \approx 1-0.01-0.00005 = 0.98995$, hence

$$\sqrt{2} \approx \frac{10}{7}(0.98995) \approx 1.41421,$$

agreeing with the true value to five decimal places, from just three terms. The key is small x: the closer x is to 0, the faster the series converges.

CAUTION Always state the interval of validity alongside an HL binomial expansion; the series is meaningless outside it. And the expansion needs the form $(1+\bullet)^n$: factor first, then expand.

At HL these expansions return as Maclaurin series in Ch. 16, where the same coefficients arise from repeated differentiation.

YOUR TURN 2.10 The Binomial Series HL

Expanding $(1+x)^n$

82. Expand each up to and including the term in x^3, stating the validity.

(a) $(1+x)^{-1}$ (b) $(1-x)^{-2}$

83. Expand each up to and including the term in x^2, stating the validity.

(a) $(1+4x)^{1/2}$ (b) $(1+x)^{1/3}$

Factorising first

84. Expand $\dfrac{1}{3-x}$ in ascending powers of x up to x^2, stating the validity.

85. Expand $(8+x)^{1/3}$ in ascending powers of x up to x^2, stating the validity.

Approximations

86. Use the expansion of $(1+x)^{1/2}$ up to the term in x^2, with $x = 0.1$, to estimate $\sqrt{1.1}$ to four decimal places.

87. This question estimates $\sqrt{3}$.

(a) Expand $(4-x)^{1/2}$ in ascending powers of x up to x^2, stating the validity.

(b) By choosing a suitable value of x, use your expansion to estimate $\sqrt{3}$ to three decimal places.

The link with geometric series

88. This question compares the binomial series with a geometric series.

(a) Use the binomial series to expand $(1-3x)^{-1}$ up to and including the term in x^3.

(b) Show that these are the first four terms of a geometric series, and state its common ratio.

(c) Explain why the binomial expansion and the geometric series both need $|x| < \frac{1}{3}$.

Concept check

89. A student expands $(1-3x)^{-2}$ as a binomial series and states that it is valid for $|x| < 1$. Identify the error, and state the correct condition.

Chapter 2 · Sequences & Series

Reflections

TOK Around 450 BCE Zeno argued that a runner can never finish a race. First the runner must reach the halfway point, then half of what remains, and so on without end. The sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$ seems to answer him, since infinitely many steps can have a finite total. Yet no finite number of steps ever reaches 1: we accept the answer because we define the sum to be the limit of the partial sums. Is that a discovery about infinity, or a decision we made so that the addition has an answer? And does the decision tell us anything about how a real runner moves?

IM The triangle of binomial coefficients carries a different name in almost every place it was found. Indian scholars met these numbers counting the patterns of short and long syllables in poetry and set them out as a staircase; Persian mathematicians used them to expand powers, and Iran still calls it the Khayyam triangle; China calls it Yang Hui's triangle, Italy Tartaglia's, and much of Europe names it after Pascal, whose treatise of 1654 came centuries after most of these. The same pattern was found independently again and again, and the names record who made it well known in each place, not always who reached it first. What does that tell you about who a mathematical result belongs to?

RW A patient takes 100 mg of a drug every 12 hours, and in each 12-hour interval the body removes half of the drug present. Just after each dose the amount in the body is 100, then $100 + 50$, then $100 + 50 + 25$ mg, and so on: the partial sums of a geometric series with $u_1 = 100$ and $r = \frac{1}{2}$. The amount rises towards $S_\infty = \dfrac{100}{1 - \frac{1}{2}} = 200$ mg and never passes it. Doctors use this steady level to choose a dose and a dosing interval, so that the drug stays above the amount that works and below the amount that harms.

EXT A geometric series with $|r| < 1$ is the simplest example of a **power series**, a sum of increasing powers of x:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}, \qquad |x| < 1.$$

This one identity produces many others. Replace x by $-x$ and integrate each term to obtain the series for $\ln(1 + x)$; replace x by $-x^2$ and integrate to obtain the series for $\arctan x$. Many further functions follow in the same way, and calculus courses often introduce power series with it.

End-of-Chapter Problems
Chapter 2: Sequences & Series

1. An arithmetic sequence has first term a and common difference d. The 5th term is 17 and the sum of the first 10 terms is 155.

(a) Write down two equations in a and d. [2]
(b) Find a and d. [2]
(c) Find the first negative term of the sequence. [2]

2. An arithmetic sequence has $u_3 = 7$ and $u_9 = 25$.

(a) Find u_1 and d. [3]
(b) Find S_{20}. [2]
(c) Find the least value of n such that $S_n > 600$. [3]

3. An arithmetic series has $u_1 = 2$ and $S_{10} = 155$.

(a) Find the common difference. [2]
(b) Find the sum $\sum_{k=11}^{20} u_k$. [4]

4. An arithmetic sequence has 20 terms. The sum of the odd-positioned terms $u_1 + u_3 + u_5 + \cdots + u_{19}$ is 160, and the sum of the even-positioned terms $u_2 + u_4 + u_6 + \cdots + u_{20}$ is 140. Find the first term u_1 and the common difference d. [6]

5. The sum of the first n terms of a series is given by $S_n = 2n^2 + 5n$.

(a) Find u_1 and u_2. [2]
(b) Show that the series is arithmetic and state the common difference. [3]
(c) Find the value of n such that $u_n = 83$. [2]

6. An AP has $u_1 = 120$ and $d = -3$.

(a) Find the first term that is negative. [3]
(b) Find the maximum value of S_n and the value(s) of n for which it occurs. [4]

7. Consider the two arithmetic sequences $4, 7, 10, \ldots$ and $3, 7, 11, \ldots$ Both sequences contain the value 7.

(a) Find the general term of each sequence. [2]
(b) Find the next three values that appear in both sequences. [4]

(c) Show that the common terms form an arithmetic sequence and state its common difference. [2]

8. A geometric sequence has first term a and common ratio r, where $r > 0$. Given $u_3 + u_4 = 12$ and $u_5 + u_6 = 3$, find a and r. [5]

9. The first three terms of a geometric sequence are k, 6, and $k + 5$.

(a) Show that $k^2 + 5k - 36 = 0$, and hence find the possible values of k. [4]
(b) For the value of k that gives a convergent series, find S_∞. [3]

10. The first, second, and fifth terms of an arithmetic sequence with first term 3 are, in that order, the first, second, and third terms of a geometric sequence.

(a) Given the arithmetic sequence is non-constant, show that its common difference is $d = 6$ and the geometric sequence has common ratio $r = 3$. [4]
(b) Determine whether the geometric series $3 + 9 + 27 + \cdots$ has a sum to infinity, giving a reason. [2]

11. A sequence is defined by $u_1 = 3$ and $u_{n+1} = 2u_n + 1$ for $n \geq 1$.

(a) Write down u_2, u_3, and u_4. [2]
(b) Let $v_n = u_n + 1$. Show that v_n is a geometric sequence. [3]
(c) Find an explicit formula for u_n. [2]

12. An arithmetic sequence has first term a and common difference d, where $a > 0$ and $d \neq 0$. Its 1st, 4th and 13th terms form a geometric sequence.

(a) Show that $3d = 2a$. [4]
(b) Given that $a = 6$, find d and write down the first four terms of the arithmetic sequence. [3]
(c) Find the common ratio of the geometric sequence. [2]
(d) Find the sum of the first 20 terms of the arithmetic sequence. [3]
(e) Find the sum of the first 8 terms of the geometric sequence. [3]

13.

(a) By expanding $(1 - r)(1 + r + r^2 + \cdots + r^{n-1})$, show that $\displaystyle\sum_{k=1}^{n} r^{k-1} = \frac{1 - r^n}{1 - r}$ for $r \neq 1$. [3]

(b) Hence find $\displaystyle\sum_{k=1}^{10} \left(\tfrac{2}{3}\right)^{k-1}$, giving your answer as an exact fraction. [3]

14.

(a) Find $\sum_{k=1}^{n}(4k-3)$ in terms of n. [3]

(b) Find the value of n such that $\sum_{k=1}^{n}(4k-3) = 780$. [3]

15. GDC Let $S_n = \sum_{k=1}^{n}\left(3k + 2(0.8)^k\right)$.

(a) Show that $S_n = \dfrac{3n(n+1)}{2} + 8\left(1 - 0.8^n\right)$. [3]

(b) Find the least value of n for which $S_n > 1000$. [2]

16. GDC A geometric series has $u_1 = 3$ and $S_\infty = 12$.

(a) Find the common ratio. [2]

(b) Show that $S_n = 12\left(1 - \left(\frac{3}{4}\right)^n\right)$. [2]

(c) Find the least value of n such that $S_n > 11.9$. [3]

17. Consider the geometric series $1 + 2x + 4x^2 + 8x^3 + \cdots$

(a) Write down the common ratio. [1]

(b) Find the values of x for which the sum to infinity exists. [2]

(c) For these values, find S_∞ as a function of x. [2]

18. A geometric sequence has all positive terms. The sum of the first two terms is 10 and the sum to infinity is 25. Find u_1 and r. [5]

19. The sum to infinity of a geometric series is $\frac{3}{2}$ times the sum of the first 4 terms. The common ratio is positive. Find the common ratio. [5]

20. The repeating decimal $0.\overline{142857} = 0.142857142857\ldots$ can be written as an infinite geometric series.

(a) Write down u_1 and the common ratio r. [2]

(b) Find the exact fractional value of $0.\overline{142857}$. [3]

21. GDC A ball is dropped from a height of 10 m. After each bounce it reaches 70% of the height it fell from.

(a) Find the height reached after the 3rd bounce. [2]

(b) Find the total vertical distance travelled by the ball before it comes to rest. [4]

(c) Find the least number of bounces after which the height reached is less than 1 cm. [3]

22. GDC Two cyclists train for the same race. In week 1 Ana rides 40 km, and each week after that she rides 8% further than in the week before. In week 1 Ben rides 60 km, and each week after that he rides 5 km further than in the week before.

(a) Find the distance each cyclist rides in week 10. Give Ana's distance to 3 significant figures. [3]
(b) Find the total distance each cyclist rides in the first 10 weeks, to the nearest km. [4]
(c) Find the first week in which Ana rides further than Ben. [3]
(d) Find the least number of weeks after which Ana's total distance is greater than Ben's total distance. [3]

23. GDC A machine is bought for \$60 000 and depreciates at 18% per year.

(a) Find its value after 4 years. [2]
(b) Find the least number of complete years until the value falls below one quarter of the purchase price. [3]

24. GDC \$8000 is invested for 6 years at 5% per year, compounded quarterly. Over the same period, inflation runs at 3% per year.

(a) Find the future value of the investment. [2]
(b) Find the real value of the investment after 6 years. [2]
(c) Comment on what your answers say about the investment's true growth. [1]

25. GDC An investment account pays 5% per year, compounded annually. A sum P is invested.

(a) Write an expression for the value of the account after n years. [1]
(b) Find the value of P if the account is worth \$20 000 after 10 years. [3]
(c) Find the number of complete years for the account to triple in value. [3]

26. GDC The value of a car decreases by the same percentage each year. A new car costs \$32 000, and after 3 years it is worth \$16 384.

(a) Show that the value falls by 20% each year. [2]
(b) Find the year in which the value first falls below \$10 000. [3]
(c) Find the total loss in value over the first 5 years. [3]

27. GDC A student can invest \$3000 in one of two accounts. Account A pays 5% per year *simple* interest; account B pays 4% per year, compounded annually.

(a) Write down an expression for the value of each account after n years. [3]
(b) Find the value of each account after 10 years. [2]
(c) Using a table or graph on your GDC, find the least whole number of years after which account B is worth more than account A. [3]

28. GDC A savings plan pays \$500 into an account at the start of each year. The account earns 4% per year, compounded annually.

(a) Show that after 3 years the total value is $500(1.04)^3 + 500(1.04)^2 + 500(1.04)$. [2]
(b) Write this as a geometric series and find the total after 10 years. [4]
(c) Find the number of complete years until the total exceeds \$8 000. [3]

29. GDC An amount P is invested. Bank A pays 4% interest compounded annually. Bank B pays 3.9% compounded monthly, so its value after t years is $P\left(1+\dfrac{0.039}{12}\right)^{12t}$.

(a) Write down an expression for the value in Bank A after t years. [1]
(b) Show that Bank B's value can be written as Pr^t, and find r to five decimal places. [3]
(c) Determine which bank gives more after 10 years, and by how much as a percentage of P. [3]
(d) Use logarithms to find how long an investment in Bank A takes to double. [2]
(e) Bank C pays 4% compounded continuously, giving $Pe^{0.04t}$. Show that Bank C always beats Bank A, and find the difference after 10 years as a percentage of P. [3]

30. The seven letters of the word FACTORS are all different. Two of them, A and O, are vowels.

(a) Find the number of arrangements of all seven letters. [1]
(b) Find the number of these arrangements that begin with a vowel and end with a consonant. [2]
(c) Find the number of these arrangements in which the two vowels are not next to each other. [3]
(d) A code is formed from 4 of the seven letters, in order, with no letter repeated. Find the number of codes that contain the letter T. [2]

31. A school committee of 6 is to be chosen from 9 teachers and 7 students.

(a) Find the number of committees containing exactly 4 students. [2]
(b) Find the number of committees containing at least 4 students. [3]

32.

(a) Show that ${}^nC_2 = \dfrac{n(n-1)}{2}$, and explain the connection with the sum $1+2+\cdots+(n-1)$. [3]
(b) Hence find n such that ${}^nC_2 = 190$. [2]

33. In the expansion of $(1+kx)^8$, the coefficient of x^2 is 112.

(a) Find the possible values of k. [3]

(b) For $k > 0$, find the coefficient of x^3. [2]

34. Consider the expansion of $\left(x + \dfrac{2}{x^2}\right)^{12}$.

(a) Find the term independent of x. [4]

(b) Find the coefficient of x^3. [2]

35. Consider the expansion of $\left(2x - \dfrac{3}{x}\right)^{10}$.

(a) Write down an expression for the general term of the expansion. [2]

(b) Find the term that is independent of x. [4]

(c) Find the coefficient of x^4. [3]

(d) Explain why the expansion contains no term in x^3. [2]

(e) By substituting a suitable value of x, find the sum of all the coefficients. [2]

(f) State how many terms the expansion has, and which powers of x occur. [2]

36.

(a) Expand $(1 + 2x)^{-2}$ in ascending powers of x up to and including the term in x^3, and state the values of x for which the expansion is valid. [4]

(b) Use your expansion with $x = 0.01$ to estimate $\dfrac{1}{1.02^2}$ to 5 decimal places. [2]

37.

(a) Show that $(4 + x)^{1/2} = 2\left(1 + \dfrac{x}{4}\right)^{1/2}$, and expand it in ascending powers of x up to the term in x^2, stating the validity. [4]

(b) Hence find an approximation to $\sqrt{4.2}$, giving four decimal places. [2]

Challenge Questions

38. GDC **When does a sequence settle?** A sequence is defined by a starting value u_1 and the rule $u_{n+1} = au_n + b$, where a and b are constants. This investigation finds, for every value of a, whether the terms approach a limit.

(a) Take $a = \frac{1}{2}$, $b = 3$ and $u_1 = 2$. Tabulate $u_1, u_2, \dots, u_6$, and state the value the terms appear to approach. [2]

(b) Let $a \neq 1$. Show that if the sequence has a limit L, then $L = aL + b$, and hence $L = \dfrac{b}{1-a}$. Check this against part (a). [2]

(c) With $a \neq 1$ and $L = \dfrac{b}{1-a}$, let $v_n = u_n - L$. Show that $v_{n+1} = av_n$, and deduce that $u_n = L + (u_1 - L)\,a^{n-1}$. [4]

(d) Suppose $u_1 \neq L$. Use part (c) to describe the long-run behaviour of $\{u_n\}$ when $0 < |a| < 1$, when $a = -1$ and when $|a| > 1$, saying in each case whether the sequence converges, and whether the terms stay on one side of L or alternate about it. Describe also what happens when $a = 1$. [4]

(e) A patient takes 100 mg of a drug every 12 hours, and by the time of the next dose the body has removed 60% of the drug present. Let u_n be the amount, in mg, just after the nth dose, so $u_1 = 100$ and $u_{n+1} = 0.4u_n + 100$. Find the long-run level, and the first dose after which the amount is within 1 mg of it. Find also the largest dose that keeps the long-run level at or below 200 mg. [4]

(f) The rule $u_{n+1} = u_n^2 + c$, with $u_1 = 0$, is not of the form above. Show that a limit L can exist only if $c \leq \frac{1}{4}$. Explore the sequence for $c = -0.5,\ 0.3,\ -1$ and -2.1, describe what you see, and state a question about this rule that could be investigated further. [3]

39. GDC **Counting multiples.** Each set of multiples below is an arithmetic sequence, so its total is an arithmetic series. Throughout, $\lfloor t \rfloor$ is the greatest integer not exceeding t; in parts (a) to (d), work with the integers from 1 to 1000.

(a) Find the sum of those divisible by 3, and the sum of those divisible by 5. [3]

(b) Explain why adding your two answers counts some integers twice, and find the sum of those divisible by both. [3]

(c) Hence find the sum of those divisible by 3 or by 5. [2]

(d) Find the sum of those divisible by 3 or by 5, but not by both. [3]

(e) Let p and q be coprime integers greater than 1 (their only common positive factor is 1), and let N be a positive integer. Show that the sum of the integers from 1 to N divisible by p or q but not both is

$$S_p + S_q - 2S_{pq}, \qquad \text{where } S_m = m\,\frac{t(t+1)}{2}, \quad t = \left\lfloor \frac{N}{m} \right\rfloor.$$

[4]

(f) Use the result of part (e) with $p = 7$, $q = 11$ and $N = 1000$. Verify your answer by summing a list on your GDC. [3]

(g) Let $T(N)$ be the sum in part (e). Show that $\dfrac{T(N)}{N^2} \to \dfrac{1}{2p} + \dfrac{1}{2q} - \dfrac{1}{pq}$ as $N \to \infty$, and evaluate this for $p = 3$, $q = 5$. Suggest how the investigation could be extended. [3]

40. **A sum that is not a sum.** You may have seen the claim that $1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$. Every term is positive, so the statement cannot mean what addition normally means. This question takes the popular argument apart.

Write $G = 1 - 1 + 1 - 1 + \cdots$ and $T = 1 + 2 + 3 + 4 + \cdots$

(a) Write down the first six partial sums of G. Explain why G has no sum in the usual sense.

[3]

(b) Bracketing the terms as $(1-1)+(1-1)+\cdots$ gives 0; bracketing them as $1-(1-1)-(1-1)-\cdots$ gives 1. State what this shows about applying the ordinary rules of algebra to such a series. [3]

(c) Applying $S_\infty = \dfrac{u_1}{1-r}$ to G with $u_1 = 1$ and $r = -1$ gives $\frac{1}{2}$. State the condition under which that formula was derived, and explain why it does not hold here. [3]

(d) Write down the nth partial sum of T and show that it increases without bound. [3]

(e) The popular argument runs: assume $G = \frac{1}{2}$; deduce $1-2+3-4+\cdots = \frac{1}{4}$; subtract this from T term by term to obtain $4+8+12+\cdots = 4T$; conclude $T - \frac{1}{4} = 4T$, so $T = -\frac{1}{12}$. Identify the step that is not valid, and say why. [4]

(f) Take the partial sums of G from part (a) and average the first n of them. Compute these averages for $n = 1$ to 6, and state what value they appear to approach. [3]

(g) Part (f) shows that $\frac{1}{2}$ is not arbitrary: it is the value G receives when "sum" is defined as the limit of these averages. Show that this averaging definition gives T no value. Then explain what a statement such as $1+2+3+\cdots = -\frac{1}{12}$ actually asserts, and why writing it with an equals sign is misleading. [3]

(h) Another definition (Abel's) replaces a series $c_0 + c_1 + c_2 + \cdots$ by $c_0 + c_1x + c_2x^2 + \cdots$ for $0 < x < 1$, and lets x approach 1 from below. Find the value this gives G. Given that $1 - 2x + 3x^2 - 4x^3 + \cdots = \dfrac{1}{(1+x)^2}$ for $0 < x < 1$, find the value it gives $1-2+3-4+\cdots$, and comment on the result in the light of part (e). Suggest how comparing definitions of "sum" could be taken further. [3]

41. GDC **How many terms for $\sqrt{2}$?** The binomial series for $(1-x)^{1/2}$ converges for $|x| < 1$, but how quickly it converges depends on x. This investigation measures the error made by stopping the series early, and uses the result to choose x well. Write

$$(1-x)^{1/2} = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots.$$

(a) Show that each of the following is true:

$$\sqrt{2} = 2\left(1-\tfrac{1}{2}\right)^{1/2}, \qquad \sqrt{2} = \tfrac{10}{7}\left(1-\tfrac{1}{50}\right)^{1/2}, \qquad \sqrt{2} = \tfrac{17}{12}\left(1-\tfrac{1}{289}\right)^{1/2}.$$

[3]

(b) Find $c_0, c_1, \ldots, c_4$. Show that $\dfrac{c_{k+1}}{c_k} = \dfrac{2k-1}{2k+2}$ for $k \geq 1$, and deduce that every coefficient after c_0 is negative and that $|c_{k+1}| < |c_k|$ for $k \geq 1$. [4]

(c) Let $0 < x < 1$ and $m \geq 1$, and let E_m be the amount by which $c_0 + c_1x + \cdots + c_mx^m$ exceeds $(1-x)^{1/2}$. Show that

$$0 < E_m < \frac{|c_{m+1}|\, x^{m+1}}{1-x}.$$

(Hint: compare the omitted terms with a geometric series) [4]

(d) For each expression in part (a), use the series up to the x^3 term to approximate $\sqrt{2}$. Give the actual error in each approximation and the bound from part (c), each to two significant figures. [4]

(e) Using $\sqrt{2} = \frac{17}{12}\left(1 - \frac{1}{289}\right)^{1/2}$, find the least m for which the bound in part (c) guarantees an error in $\sqrt{2}$ below 10^{-12}. [3]

(f) The fraction $\frac{17}{12}$ satisfies $p^2 - 2q^2 = 1$. Show that whenever positive integers p and q satisfy $p^2 - 2q^2 = 1$, $\sqrt{2} = \dfrac{p}{q}\left(1 - \dfrac{1}{p^2}\right)^{1/2}$. Find the next such fraction after $\frac{17}{12}$, and estimate how many correct decimal places each extra term of the series gives for it, compared with $x = \frac{1}{2}$ and $x = \frac{1}{289}$. Suggest a direction in which this could be investigated further. [3]