NUMBER & ALGEBRA

CHAPTER 1

Exponents & Logarithms

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SYLLABUS 1.1 1.5 1.7 2.9

Japan lies in one of the world's most active seismic zones. Suppose a seismologist asks you to plot two recordings on one axis. The first has an amplitude (the height of the wave) of 10^4. The second is 10^9, a hundred thousand times larger.

An ordinary scale, with evenly spaced marks, cannot show both. If the marks are close enough together to show the first, the second lies far beyond the page. If the scale is long enough for the second, the first sits so close to zero that you cannot see it.

So let equal spaces along the axis mean multiplication by 10 rather than addition of 1. The scale then reads $10^0, 10^1, 10^2$, and the two recordings, with exponents 4 and 9, sit five equal steps apart. That exponent is the **logarithm** of the amplitude, and it is the idea this chapter is built on.

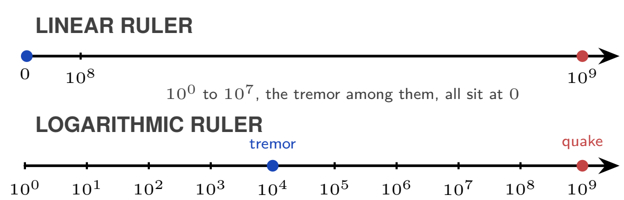

Figure 1.1 A logarithmic scale separates values that collapse near zero on a linear scale.

Earthquake magnitude works this way, and so do pH, decibels, and the brightness of stars. Writing a quantity as a power of ten turns multiplication into addition, so a range too wide to plot fits on an axis you can read. By the end of this chapter you will be able to use the laws of exponents and standard form, move between exponential and logarithmic form, apply the laws of logarithms and change the base of a logarithm, and solve equations with the unknown in the exponent.

1.1 What an Exponent Counts

An exponent, also called a *power* or an *index*, is shorthand for repeated multiplication. In a^n, a is the **base** and n is the **exponent**. When n is a positive integer,

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}}.$$

For example, $a^3 = a \cdot a \cdot a$.

When you multiply powers with the same base, it is tempting to multiply the exponents. Writing out the factors shows what actually happens:

$$a^3 \cdot a^2 = (a \cdot a \cdot a)(a \cdot a) = a^5.$$

There are five factors of a, not six. Since the exponent counts how many times the base appears as a factor, the exponents are **added**:

$$a^m \cdot a^n = a^{m+n}.$$

Division cancels common factors, so the exponents are **subtracted**:

$$\frac{a^5}{a^2} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a} = a^3 = a^{5-2}.$$

Raising a power to another power repeats the whole power, so the exponents are **multiplied**:

$$(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = a^6 = a^{2\times 3}.$$

A power outside a bracket applies to every factor inside:

$$(ab)^n = a^n b^n, \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

The same laws extend naturally to zero and negative exponents. For $a \neq 0$, the subtraction rule gives $a^3/a^3 = a^{3-3} = a^0$; since any nonzero number divided by itself is 1, this forces $a^0 = 1$. Continuing the same pattern gives $a^{-n} = 1/a^n$. These values are not imposed on us; they are the only ones that keep the laws consistent. The main laws are collected below.

DEF For $a, b \neq 0$ and $m, n \in \mathbb{Z}$:

$a^m \cdot a^n = a^{m+n}$	product of powers: add the exponents
$\dfrac{a^m}{a^n} = a^{m-n}$	quotient of powers: subtract the exponents
$(a^m)^n = a^{mn}$	power of a power: multiply the exponents
$(ab)^n = a^n b^n$	power of a product: raise each factor
$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	power of a quotient: raise top and bottom
$a^0 = 1$	zero exponent: gives 1
$a^{-n} = \dfrac{1}{a^n}$	negative exponent: take the reciprocal

CAUTION Do not confuse $(a^m)^n$ with a^{m^n}. In $(a^m)^n$, the power of a power, you multiply the exponents. In a^{m^n} you start at the top: first work out m^n, then raise a to that result. The two are usually not equal: $(2^3)^2 = 2^6 = 64$, but $2^{3^2} = 2^9 = 512$.

CAUTION A negative exponent means *reciprocal*, not *negative value*. $2^{-3} = \frac{1}{8}$, not -8.

1.1.1 Worked examples

How do these rules combine when a problem has several operations at once?

Worked through. To simplify $\frac{10c^7}{5c^3}$, split the coefficient and the variable and handle them separately:

$$\frac{10c^7}{5c^3} = \frac{10}{5} \cdot c^{7-3} = 2c^4.$$

Apply one rule at a time.

EXAMPLE 1.1

Simplify $(2x^2y^{-4})^5$, giving your answer with positive exponents. It contains several operations: a bracket raised to a power and a negative exponent. Raise each part in turn: the 2 to the 5th, x^2 to the 5th, and y^{-4} to the 5th.

$$(2x^2y^{-4})^5 = 32x^{10}y^{-20} = \frac{32x^{10}}{y^{20}}.$$

CAUTION The bracket in $(2x^2)^3$ applies to everything inside it, including the 2. $(2x^2)^3 = 8x^6$, not $2x^6$.

1.1.2 Beyond counting factors

The idea of a^n as repeated multiplication only makes sense when n is a positive integer. The laws above already fixed $a^0 = 1$ and $a^{-n} = 1/a^n$. A fractional exponent such as $3^{1/2}$ needs the same treatment: we choose its value so that the laws keep working, and the next section shows that this gives $3^{1/2} = \sqrt{3}$.

1.1.3 Surds and fractional exponents

What is $9^{1/2}$? Since $\frac{1}{2}$ is not an integer, repeated multiplication cannot answer this directly. But the power-of-a-power law gives

$$\left(9^{1/2}\right)^2 = 9^{(1/2)\times 2} = 9,$$

so $9^{1/2}$ must be the non-negative number whose square is 9:

$$9^{1/2} = \sqrt{9} = 3.$$

Notice the value is 3, not ± 3: the symbol $\sqrt{9}$ means the **principal** (non-negative) square root.

DEF For $a > 0$, $n \in \mathbb{Z}^+$ and $m \in \mathbb{Z}$:

$$a^{1/n} = \sqrt[n]{a} \qquad \text{unit fraction: the denominator is the root}$$

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m} \qquad \text{in general: root then power, in either order}$$

When n is even the symbol $\sqrt[n]{a}$ means the **positive** root, exactly as $\sqrt{9}$ means 3 and not ± 3.

A useful way to remember this:

$$\boxed{\text{denominator } n \longrightarrow \text{root}, \qquad \text{numerator } m \longrightarrow \text{power}}$$

These are not new laws; they are the meanings chosen so that the familiar exponent laws keep holding.

USING fractional exponents

For $a^{m/n}$, it is usually easier to take the root first, then apply the power:

$$64^{2/3} = \left(\sqrt[3]{64}\right)^2 = 4^2 = 16.$$

Either order gives the same result, $64^{2/3} = \sqrt[3]{64^2} = 16$, but taking the root first keeps the numbers small.

The same laws run backwards. If the unknown carries a fractional exponent, raise both sides to the reciprocal of that exponent: the two multiply to 1, and the power-of-a-power law leaves the unknown on its own. The definition above covers positive bases only, so this move is safe when the question restricts the unknown to positive values. If a question allows negative values and reads $x^{2/5}$ as $(\sqrt[5]{x})^2$, the even power discards the sign: $x^{2/5} = 4$ gives $\sqrt[5]{x} = \pm 2$, so $x = \pm 32$, while raising both sides to $\frac{5}{2}$ finds only 32. Check the restriction on x before counting solutions.

EXAMPLE 1.2

Solve $x^{3/4} = 125$ for $x > 0$.

The exponent $\frac{3}{4}$ is undone by $\frac{4}{3}$, since $\frac{3}{4} \times \frac{4}{3} = 1$. Raise both sides to that power:

$$\left(x^{3/4}\right)^{4/3} = 125^{4/3} \implies x^1 = \left(\sqrt[3]{125}\right)^4 = 5^4 = 625.$$

Check by substituting: $625^{3/4} = \left(\sqrt[4]{625}\right)^3 = 5^3 = 125$. The condition $x > 0$ is what allows the fractional exponent in the first place, since $x^{3/4}$ means $(\sqrt[4]{x})^3$ and the fourth root needs a non-negative input.

WORKING with surds

A **surd** is an irrational root left in exact form, such as $\sqrt{2}$ or $\sqrt[3]{5}$, rather than rounded to a decimal. On a non-calculator paper, surds should be kept exact and simplified using the root laws

$$\sqrt{ab} = \sqrt{a}\,\sqrt{b}, \qquad \sqrt{\tfrac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}},$$

valid for $a, b \geq 0$ (with $b > 0$ in the quotient). Two techniques cover most questions:

1. **Simplify** a surd by extracting its largest perfect-square factor.
2. **Combine** only like surds, the terms that have the same value under the root.

Two surds that look unrelated often become like surds once each has been simplified, so always simplify before trying to combine.

CAUTION The root laws hold for products and quotients only. $\sqrt{a} + \sqrt{b}$ is not $\sqrt{a+b}$, and

$\sqrt{a}-\sqrt{b}$ is not $\sqrt{a-b}$, so surds can be combined only when they are like surds.

Worked through. To simplify $\sqrt{147}-\sqrt{48}$, take the largest perfect-square factor out of each root:

$$\sqrt{147}=\sqrt{49\cdot 3}=7\sqrt{3}, \qquad \sqrt{48}=\sqrt{16\cdot 3}=4\sqrt{3}.$$

Both terms now carry $\sqrt{3}$, so they are like surds and only their coefficients subtract:

$$7\sqrt{3}-4\sqrt{3}=3\sqrt{3}.$$

The root laws cover products and quotients but never sums or differences, so $\sqrt{147}-\sqrt{48}$ is not $\sqrt{99}$: one is $3\sqrt{3}\approx 5.20$, the other about 9.95.

1.1.4 Rationalising the denominator

An exact answer is usually written without a surd in the denominator; removing it is called **rationalising the denominator**.

- For a single surd, multiply top and bottom by that surd.
- For a sum or difference, multiply by its **conjugate**, formed by changing the sign between the terms.

The conjugate works because $(a+\sqrt{b})(a-\sqrt{b})=a^2-b$ is a difference of two squares, so the surd drops out of the denominator.

Worked through. To rationalise $\dfrac{1}{\sqrt{3}}$ and $\dfrac{6}{2+\sqrt{3}}$, take each in turn. For the first, multiply by $\dfrac{\sqrt{3}}{\sqrt{3}}$:

$$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}.$$

For the second, multiply by the conjugate $2-\sqrt{3}$:

$$\frac{6}{2+\sqrt{3}}=\frac{6(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}=\frac{6(2-\sqrt{3})}{4-3}=12-6\sqrt{3}.$$

The conjugate works because the product is a difference of two squares, $(2+\sqrt{3})(2-\sqrt{3})=2^2-(\sqrt{3})^2=1$, so the surd leaves the denominator.

TIP In the real numbers, $(-4)^{1/2}$ is undefined: no real number squares to -4. In general, the radicand, the number inside the root, must be non-negative for an even root to have a real value. Square roots of negative numbers are complex numbers, which are introduced in Chapter 17.

1.1.5 Exponents as continuous growth

There is another way to read a non-integer exponent. Suppose a quantity doubles once in every unit of time, so that 2^t is its growth factor after time t. Then 2^3 is three complete doublings, while

$$2^{1/2} = \sqrt{2}$$

is the growth factor halfway through one doubling period. Because the growth is exponential, "halfway" refers to the time, not to half the final increase. This reading extends to every real value of t, whether negative, fractional, or irrational:

$$2^{-3}, \qquad 2^{0}, \qquad 2^{1/2}, \qquad 2^{\sqrt{2}}.$$

Read as growth over time t, each has a plain meaning. $2^0 = 1$ is no growth at all: at time zero the population is unchanged, one hundred per cent of where it started. $2^{-3} = \frac{1}{8}$ looks backwards in time, to three periods before now (three seconds ago, if one doubling takes a second), when the population was an eighth of its present size. $2^{1/2} = \sqrt{2}$ is the factor after half a period, and $2^{\sqrt{2}}$ the factor after an irrational stretch of time. For any positive base a, the expression a^x has a meaning for every real exponent x, which is where the exponential function of Chapter 5 begins.

1.1.6 The number *e*

Suppose an account pays 100% interest in a year. Paid once, at the end, an amount of 1 becomes 2. Now suppose the same 100% arrives in two instalments of 50%. After six months the balance is $1 \times 1.5 = 1.5$, and the second 50% is taken on 1.5 rather than on 1, so the year ends at $1.5 \times 1.5 = 2.25$. Splitting the same total rate into more instalments gives more, because an amount added early is itself subject to all the growth that follows.

That leads to a question with a surprising answer. If a finer split always gives more, does the year-end amount grow without limit? In n equal steps the starting amount is multiplied by $\left(1 + \frac{1}{n}\right)^n$, so the question is what that expression does as n increases.

instalments in the year, n	**amount after one year**
1 (once, at the end)	2.00000
2 (every six months)	2.25000
12 (monthly)	2.61304
365 (daily)	2.71457
8760 (hourly)	2.71813
$n \to \infty$ (continuous)	$2.71828\ldots = e$

It does not grow without limit. Each extra split adds less than the split before it, because the money it adds arrives later in the year and so has less of the year left in which to earn anything itself. Going from yearly to six-monthly gains 0.25; going from daily to hourly gains about 0.0036. The totals close in on one value, $2.71828\ldots$, and that number is called e. Like π, it is irrational: its decimal expansion neither ends nor repeats.

So e answers one specific question: what does 100% growth come to when it acts continuously instead of in steps? Other rates are handled in the exponent. A continuous rate k acting for time t multiplies the starting amount by e^{kt}. This model describes radioactive decay, and it is a common first model for population growth and for the gap between a cooling object's temperature and its surroundings. Chapter 13 gives a deeper reason for e: the function e^x grows at a rate equal to its own value.

EXT Formally, $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. Each n is a number of compounding steps per year, so letting n grow without bound is what "continuously" means. Limits are defined properly in Ch. 13.

YOUR TURN 1.1 What an Exponent Counts

Laws of exponents

1. Simplify each of the following, giving your answers with positive exponents.

(a) $\dfrac{6c^9}{2c^4}$ (b) $(3x^2y^{-3})^4$

(c) $\dfrac{12a^5b^3}{4a^2b^{-1}}$ (d) $\dfrac{(2x^3)^2\cdot x^{-4}}{4x^2}$

2. Simplify each of the following.

(a) $\dfrac{4x^3-8x^2}{2x}$ (b) $\dfrac{16a^5b^2-12ab^4}{4ab^2}$

3. Show that $\dfrac{(a^2b^{-1})^3}{a^{-3}b^2} = \dfrac{a^9}{b^5}$.

4. Evaluate each pair of numbers without a calculator, and state whether the two numbers are equal.

(a) $(2^2)^3$ and 2^{2^3} (b) $(3^2)^2$ and 3^{2^2}

Fractional exponents

5. Evaluate each of the following.

(a) $27^{2/3}$ (b) $81^{-3/4}$

(c) $\left(\dfrac{8}{27}\right)^{-2/3}$ (d) $32^{-2/5}$

6. Write each of the following in the form x^n.

(a) $\sqrt[4]{x^3}$ (b) $\dfrac{1}{\sqrt[3]{x^2}}$

(c) $\dfrac{\sqrt{x^5}}{x^2}$ (d) $\dfrac{x^2\cdot\sqrt{x}}{x^{3/2}}$

Equations with a fractional exponent

7. Solve each equation for $x > 0$.

(a) $x^{3/2} = 64$ (b) $x^{2/3} = 9$
(c) $x^{-1/2} = \frac{1}{4}$ (d) $x^{5/2} = 32$

8. In this question x may be any real number, and $x^{m/n}$ means $\left(\sqrt[n]{x}\right)^m$ with n odd. Solve each equation, giving every real solution.

(a) $x^{2/5} = 4$ (b) $x^{4/3} = 81$

Surds

9. Simplify each of the following.

(a) $\sqrt{50} - \sqrt{8}$ (b) $\sqrt{12} + \sqrt{27}$
(c) $\sqrt{18} + \sqrt{32} - \sqrt{2}$ (d) $\sqrt{75} - 2\sqrt{12}$

Rationalising the denominator

10. Rationalise the denominator of each expression.

(a) $\dfrac{4}{3-\sqrt{5}}$, in the form $a + b\sqrt{5}$ (b) $\dfrac{6}{\sqrt{3}}$
(c) $\dfrac{1}{\sqrt{7}+\sqrt{5}}$ (d) $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$

11. Rationalise $\dfrac{4+\sqrt{15}}{4-\sqrt{15}}$, giving your answer in the form $a + b\sqrt{15}$.

Concept check

12. A student writes $\sqrt{9+16} = \sqrt{9} + \sqrt{16} = 7$.

(a) Identify the error, and find the correct value.
(b) State the law for square roots that does hold, and use it to simplify $\sqrt{9 \times 16}$.

1.2 Standard Form

The mass of a proton, written out in full, is

$$0.00000000000000000000000000167 \text{ kg},$$

that is, 26 zeros after the decimal point before the first significant figure. The mass of the Sun runs to the opposite extreme,

$$1{,}990{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000 \text{ kg},$$

a 1 followed by 30 digits. Both numbers are quite difficult to write, let alone compare or calculate with.

There is a shorter way to write both. Instead of all the digits, record *the power of 10* that places the decimal point where it belongs. Mathematicians call the result **standard form** (also known as scientific notation).

DEF A number is in **standard form** when written as $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

The proton becomes 1.67×10^{-27} kg. The Sun becomes 1.99×10^{30} kg. The exponent k tells you immediately whether you are dealing with something microscopic or cosmic.

1.2.1 Calculating with standard form

Multiplication and division follow straight from the exponent laws. For a product, multiply the coefficients and add the powers of 10; for a quotient, divide the coefficients and subtract the powers:

$$(a \times 10^m)(b \times 10^n) = (a \times b) \times 10^{m+n}.$$

One check matters afterwards. If the new coefficient falls outside the range $1 \le a < 10$, shift the decimal point and adjust the power. For example, $(5 \times 10^3)(4 \times 10^2) = 20 \times 10^5 = 2 \times 10^6$: the coefficient 20 becomes 2, and the exponent rises by one to account for the extra factor of 10.

Addition and subtraction need one more move first. Just as you cannot add metres and kilometres without converting, you cannot add two numbers in standard form until they share the same power of 10. Rewrite the one with the smaller exponent so the powers match, then add the coefficients.

EXAMPLE 1.3

Calculate $(3.5 \times 10^4) \times (2 \times 10^{-7})$.

Group and multiply:

$$(3.5 \times 2) \times 10^{4+(-7)} = 7 \times 10^{-3}.$$

EXAMPLE 1.4

Calculate $(4.8 \times 10^5) + (3.2 \times 10^4)$.

Rewrite 3.2×10^4 as 0.32×10^5 so both terms share the power 10^5:

$$4.8 \times 10^5 + 0.32 \times 10^5 = 5.12 \times 10^5.$$

CAUTION In addition and subtraction, do **not** add the exponents. Only multiplication and division combine powers of 10.

CAUTION Calculator display notation is not acceptable in written answers. If your graphic display calculator (GDC) shows 5.2E30, you must write 5.2×10^{30}.

YOUR TURN 1.2 Standard Form

Converting between forms

13. Write 0.000047 in standard form.

14. Write each number in ordinary form.

(a) 3.2×10^5

(b) 8.06×10^{-4}

(c) 1.5×10^7

(d) 9×10^{-1}

Calculating in standard form

15. Calculate $(2.4 \times 10^3) \times (3 \times 10^{-5})$, giving your answer in standard form.

16. Calculate $\dfrac{6.6 \times 10^8}{2.2 \times 10^3}$, giving your answer in standard form.

17. Calculate each of the following, giving your answer in standard form.

(a) $(5.1 \times 10^4) + (3.9 \times 10^3)$

(b) $(7.2 \times 10^{-3}) - (5 \times 10^{-4})$

(c) $(2.5 \times 10^6) + (7.5 \times 10^6)$

(d) $(9.4 \times 10^{-2}) + (6 \times 10^{-2})$

Using a calculator

18. A GDC shows each result below in its display notation. Write each one in standard form, as it must appear in a written answer.

(a) 6.02E23

(b) 4.5E-8

19. GDC The mass of an electron is 9.11×10^{-31} kg. The mass of a proton is 1.67×10^{-27} kg. Find how many times heavier a proton is than an electron. Give your answer in standard form to 3 significant figures.

Concept check

20. A student writes $(3 \times 10^4) + (2 \times 10^3) = 5 \times 10^7$. Identify the error, and give the correct answer in standard form.

1.3 The Logarithm: an Exponent Reversed

Many familiar operations come in inverse pairs: addition with subtraction, multiplication with division. So what is the inverse of exponentiation? If $a^x = b$, and you know a and b, how do you find x? That is what a logarithm does. It is not a new idea: it is the exponential equation $a^x = b$ rearranged to make the exponent x the subject.

DEF · IB For $a > 0$, $a \neq 1$, and $b > 0$:

$$a^x = b \iff x = \log_a b$$

The **common logarithm** $(\log x)$ uses base 10. The **natural logarithm** $(\ln x)$ uses base $e \approx 2.718$, the continuous-growth constant met earlier in this chapter.

The condition $b > 0$ in that definition is not a technicality. A positive base raised to any real power gives a positive result: $10^2 = 100$, $10^0 = 1$, $10^{-3} = 0.001$. No exponent turns 10 into -5, and no exponent turns 10 into 0. So $\log(-5)$ and $\log 0$ do not exist. **Whatever appears inside a logarithm must be positive.** The same holds for every base: **$\log_a x$ is defined only when $x > 0$.**

The base carries conditions of its own. It must be positive: a negative base has no consistent meaning for fractional exponents, and base 0 gives only 0 or undefined powers; and it cannot be 1, because every power of 1 is 1 and no exponent would give any other value.

The word *logarithm* comes from the Greek *logos* (ratio) and *arithmos* (number).

The graph of $y = \log_a x$ is the mirror image of $y = a^x$ across the line $y = x$, with a vertical asymptote at $x = 0$. These graphs and their transformations are developed in Chapter 5 (Functions).

Because the exponential and logarithm are inverses, each undoes what the other does:

KEY For base 10 and base e:

$\log(10^x) = x$	base 10: true for every real x
$10^{\log x} = x$	the same pair reversed, but only for $x > 0$, since $\log x$ is undefined otherwise
$\ln(e^x) = x$	base e: true for every real x
$e^{\ln x} = x$	reversed, and again only for $x > 0$

The formula booklet prints the general statement $\log_a a^x = x = a^{\log_a x}$; these four lines are its base-10 and base-e cases.

Two special values follow at once from the definition, and both are used often: since $a^1 = a$ and $a^0 = 1$,

KEY For any base $a > 0$, $a \neq 1$:

$\log_a a = 1$	since $a^1 = a$
$\log_a 1 = 0$	since $a^0 = 1$

1.3.1 Working with logarithms

To evaluate a logarithm by hand, reverse the exponent question: $\log_a b$ is defined as the exponent that turns a into b. Read $\log_a b$ as **"a to what power gives b?"** and in many cases the answer is immediate.

Worked through. To find $\log_3 81$, ask the inverse question: 3 to what power gives 81? You know $3^4 = 81$, so $\log_3 81 = 4$. The logarithm is just the exponent.

EXAMPLE 1.5

Solve $\log_{10}(y + 1) = 3$.

The equation says: the exponent you need to get $(y + 1)$ from base 10 is 3. So $y + 1 = 10^3 = 1000$, giving $y = 999$.

EXAMPLE 1.6

Without a calculator, find $e^{3\ln 2}$.

Think of $e^{\ln 2}$ first: the exponential and natural log cancel, giving 2. So $e^{3\ln 2} = (e^{\ln 2})^3 = 2^3 = 8$.

CAUTION Solving an equation with a logarithm in it can produce a value that makes an argument negative or zero. Such a value is not a solution, however correct the algebra was. Substitute every answer back into the original equation and keep only those that leave each argument positive.

YOUR TURN 1.3 The Logarithm: an Exponent Reversed

Switching between forms

21. Rewrite each statement in the other form.

(a) Write $3^4 = 81$ in logarithmic form.

(b) Write $\log_5 x = 3$ in exponential form, and hence find x.

Evaluating logarithms

22. Evaluate each of the following without a calculator.

(a) $\log_2 32$

(b) $\log_3 \frac{1}{9}$

(c) $e^{2\ln 3}$

(d) $\ln e^5$

23. Evaluate each of the following without a calculator.

(a) $\log 10\,000$

(b) $\log 0.001$

(c) $\ln \frac{1}{\sqrt{e}}$

(d) $10^{\log 5+1}$

24. State whether each expression is defined. If it is, write down its value.

(a) $\log_5 125$

(b) $\log_5 1$

(c) $\log_5 0$

(d) $\log_5(-25)$

(e) $\log_1 7$

(f) $\log_{1/2} 8$

Logarithm and exponential as inverses

25. Each statement is either true for every value allowed by the definition, or not. State which, and correct those that are not.

(a) $\log_a(a^x) = x$ for every real x

(b) $a^{\log_a x} = x$ for every real x

(c) $\log_a 1 = 0$ for every valid base a

(d) $\log_a 0 = 0$ for every valid base a

Solving logarithmic equations

26. Solve each equation. Reject any value the definition does not allow.

(a) $\log_4(x+1) = 2$

(b) $\log_3(2x-1) = 4$

(c) $\log_2(x^2 - 3x) = 2$

Concept check

27. A student says: "$\log_{10} x = -2$ *has no solution, because a logarithm cannot be negative.*" Explain the error, and solve the equation.

1.4 The Laws of Logarithms

A single logarithm is just a number. What makes logarithms worth having is that they trade hard arithmetic for easy arithmetic: multiplication becomes addition, division becomes subtraction, and a power becomes a coefficient. The last of those is the one this chapter is heading towards, because it is what brings an unknown down out of an exponent.

The laws work because a logarithm is an exponent. Multiplying two powers with the same base adds the exponents, $a^p \cdot a^q = a^{p+q}$, and reading that sentence in terms of logarithms is already the first law: multiplication becomes addition.

The first three laws below are the three exponent laws restated for logarithms. A fourth, change of base, is a different kind of rule: it rewrites any logarithm as a ratio of two logarithms in a base of your choice.

KEY · IB For $a, b, x, y > 0$ with $a, b \neq 1$:

$\log_a(xy) = \log_a x + \log_a y$	product: a product becomes a sum
$\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$	quotient: a quotient becomes a difference
$\log_a(x^m) = m\log_a x$	power: an exponent becomes a coefficient
$\log_a x = \dfrac{\log_b x}{\log_b a}$	change of base: rewrite in any base you can compute

All three proofs share one setup. Write $x = a^p$ and $y = a^q$, so that $p = \log_a x$ and $q = \log_a y$ by definition. Each law is then a one-line consequence of the matching exponent law:

- **Product.** $xy = a^p \cdot a^q = a^{p+q}$, so $\log_a(xy) = p + q = \log_a x + \log_a y$.
- **Quotient.** $\dfrac{x}{y} = a^{p-q}$, so $\log_a\left(\dfrac{x}{y}\right) = p - q = \log_a x - \log_a y$.
- **Power.** $x^m = (a^p)^m = a^{mp}$, so $\log_a(x^m) = mp = m\log_a x$.

The third law is the one used to solve exponential equations. When the unknown is in an exponent, as in b^x, the power law gives $\log_a(b^x) = x\log_a b$, which moves the unknown into a coefficient. This often converts an exponential equation into a simpler algebraic equation.

1.4.1 Change of base

What do you do when your GDC only has $\log$ (base 10) and $\ln$ (base e), but you need a logarithm in a different base, or you want to rewrite a base-10 logarithm in terms of base e? The change-of-base law above rewrites any logarithm as a ratio of two logarithms you can compute. In practice you take base e, which turns the law into $\log_a x = \dfrac{\ln x}{\ln a}$.

To prove it, let $\log_a x = y$, so $x = a^y$. Take $\ln$ of both sides: $\ln x = y \ln a$. Solve for y.

Worked through. To evaluate $\log_3 5$ on a GDC, convert to natural logarithms:

$$\log_3 5 = \frac{\ln 5}{\ln 3} \approx \frac{1.609}{1.099} \approx 1.46.$$

One case of the law is worth carrying separately. Take the argument to be b and compute in base b; since $\log_b b = 1$, the numerator disappears:

$$\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}.$$

KEY For $a, b > 0$ with $a, b \neq 1$:

$$\log_a b = \frac{1}{\log_b a}.$$

The formula booklet does not print this, so quote it with the change-of-base step above, or work from change of base directly.

Swapping a base and an argument is the move to reach for when the unknown sits in the base, because it carries that unknown into an argument, where the ordinary laws apply. Often no swap is needed at all: a logarithm with an unknown base is still an exponent question read backwards.

EXAMPLE 1.7

Solve $\log_x 81 = 4$.

The statement says that x raised to the power 4 gives 81, so

$$x^4 = 81 \implies x = 3 \quad \text{or} \quad x = -3.$$

A base must satisfy $x > 0$ and $x \neq 1$. That rules out $x = -3$, leaving $x = 3$.

An unknown base is where those two conditions matter: the algebra alone offers two roots, and only the definition of a logarithm removes one.

1.4.2 When one base is a power of another

Questions often mix bases that are powers of one another: $\log_2 x$ alongside $\log_4 x$, or $\log_2 x$ alongside $\log_8 x$. Change of base rewrites them all in one base, and for this particular case the result is short enough to know on sight.

Change $\log_{a^2} x$ to base a:

$$\log_{a^2} x = \frac{\log_a x}{\log_a a^2} = \frac{\log_a x}{2} = \tfrac{1}{2}\log_a x,$$

since $\log_a a^2 = 2\log_a a = 2$ by the power law. Squaring the base halves the logarithm. The same steps with a^n in place of a^2 give $\log_a a^n = n$, so for $a > 0$, $a \neq 1$, $x > 0$ and any $n \neq 0$,

$$\log_{a^n} x = \frac{1}{n}\log_a x.$$

This result is not in the formula booklet. Show the change-of-base step that gives it.

Worked through. To solve $\log_2 x + \log_4 x = 6$, notice that the bases are 2 and $4 = 2^2$, so rewrite the second one in base 2: $\log_4 x = \frac{1}{2}\log_2 x$. Writing $t = \log_2 x$, the equation becomes

$$t + \tfrac{1}{2}t = 6 \implies \tfrac{3}{2}t = 6 \implies t = 4,$$

so $x = 2^4 = 16$. Check: $\log_2 16 = 4$ and $\log_4 16 = 2$, and these sum to 6.

1.4.3 Applying the laws

These laws can be used in either direction: they split one logarithm into several, or gather several back into one. Both directions are needed, in simplifying expressions and in solving equations.

Worked through. To express $\log_5\left(\frac{a^3}{b}\right)$ in terms of $p = \log_5 a$ and $q = \log_5 b$, apply the quotient law, then the power law:

$$\log_5\left(\frac{a^3}{b}\right) = \log_5 a^3 - \log_5 b = 3\log_5 a - \log_5 b = 3p - q.$$

EXAMPLE 1.8

Write $2\ln a + 6\ln b$ as a single logarithm.

Use the power law in reverse first, then the product law:

$$2\ln a + 6\ln b = \ln a^2 + \ln b^6 = \ln(a^2b^6).$$

EXAMPLE 1.9

Solve $\log_2 x + \log_2(x-2) = 3$.

The two logs on the left can combine into one product:

$$\log_2[x(x-2)] = 3 \implies x(x-2) = 2^3 = 8.$$

$$x^2 - 2x - 8 = 0 \implies (x-4)(x+2) = 0.$$

So $x = 4$ or $x = -2$. But $\log_2(-2)$ does not exist, so reject $x = -2$. The answer is $x = 4$. Always check domain when logarithms are involved. It is easy to generate solutions that look valid algebraically but are meaningless.

CAUTION $\log(x+y) \neq \log x + \log y$. The product law is about $\log(xy)$, not a sum inside the log.

YOUR TURN 1.4 The Laws of Logarithms

Combining and expanding logarithms

28. Write each expression as a single logarithm.

(a) $\log_3 5 + \log_3 7$

(b) $\log_2 24 - \log_2 3$, and evaluate it

(c) $3\log_a x + \log_a y$

(d) $2\ln a - 3\ln b + \ln c$

29. Express each logarithm in terms of the quantities given.

(a) $\log_5\left(\dfrac{a^2}{b}\right)$, in terms of $\log_5 a$ and $\log_5 b$

(b) $\log_3 18$, given $p = \log_3 2$

(c) $\log_{10} 72$, given $\log_{10} 2 = p$ and $\log_{10} 3 = q$

(d) $\log_5 75$, given $\log_5 3 = 0.682$

30. Show that $\log_a(x^2-1) - \log_a(x+1) = \log_a(x-1)$, for $x > 1$.

Change of base

31. Use the change of base formula in each of the following.

(a) Write $\log_4 9$ in terms of natural logarithms.

(b) Express $\log_8 x$ in terms of $\log_2 x$.

(c) Find the exact value of $\log_4 8$.

(d) Show that $\log_{1/2} x = -\log_2 x$.

(e) Show that $\log_a b \times \log_b(a^2) = 2$, for $a, b > 0$ and $a, b \neq 1$.

32. GDC Evaluate $\log_3 50$ to 3 significant figures.

Solving equations with the laws

33. Solve each equation, using a law of logarithms first. Reject any value the logarithms do not allow.

(a) $\log_3 x + \log_3(x+6) = 3$
(b) $\log_2(x+3) - \log_2(x-1) = 2$
(c) $2\log_5 x = \log_5 9 + \log_5 4$

Bases that are powers of one another

34. Solve each equation by writing every logarithm in the same base.

(a) $\log_3 x - \log_9 x = 1$
(b) $\log_2 x + \log_8 x = 8$
(c) $\log_4 x = \log_2(x-2)$

An unknown base

35. Solve each equation for the base x. Reject any value a base does not allow.

(a) $\log_x 49 = 2$
(b) $\log_x 8 + \log_x 2 = 4$
(c) $\log_x(x+6) = 2$

Concept check

36. A student writes $\log_3(x+9) = \log_3 x + \log_3 9 = \log_3 x + 2$.

(a) Identify the law of logarithms the student has misremembered, and state it correctly.
(b) By taking $x = 18$, show that the two sides are not equal.

37. A student writes $\dfrac{\log_2 32}{\log_2 4} = \log_2 8 = 3$. Explain the error, and find the correct value.

1.5 Solving Exponential Equations

In an equation such as $5^x = 30$, ordinary algebra cannot isolate the unknown. Adding, multiplying or factorising does not isolate the x, because x is not being combined with anything: it is counting how many times the base is used. The logarithm is what brings it down, and once it is down the equation is one you already know how to solve. This section starts with equations that need no logarithm, then uses the logarithm on harder cases.

1.5.1 Same base first

Is there ever a way to solve an exponential equation without using a logarithm at all? Before anything else, look for a common base. If both sides can be written as powers of the same number, the exponents must be equal, provided the base is positive and not 1: $a^p = a^q$

forces $p = q$ for $a > 0,\ a \neq 1$. When it is available, this is usually quicker than a logarithm, and it suits non-calculator papers.

Worked through. To solve $\left(\frac{1}{3}\right)^x = 9^{x+1}$, notice that both sides are powers of 3: $\frac{1}{3} = 3^{-1}$ and $9 = 3^2$. Rewrite and equate exponents:

$$3^{-x} = 3^{2(x+1)} \implies -x = 2x + 2 \implies x = -\tfrac{2}{3}.$$

No logarithm was needed. Check for a shared base before anything else; look for common bases such as $2, 4, 8$ and $3, 9, 27$.

1.5.2 The general strategy

When no common base exists, take a logarithm of both sides, use $\log(x^m) = m \log x$ to bring the exponent down, then solve what is now a linear or quadratic equation. The choice of which logarithm to use ($\log$ or $\ln$) does not matter: pick whichever gives cleaner arithmetic.

Worked through. To solve $5^x = 30$, take $\ln$ of both sides:

$$x \ln 5 = \ln 30 \implies x = \frac{\ln 30}{\ln 5} \approx 2.11.$$

Or equivalently, $x = \log_5 30$.

EXAMPLE 1.10

Solve $7^{x-2} = 5^{x+3}$ exactly.

With different bases on each side, take $\ln$ of both sides and use the power law:

$$(x-2)\ln 7 = (x+3)\ln 5.$$

Collect x on one side:

$$x(\ln 7 - \ln 5) = 3\ln 5 + 2\ln 7 \implies x = \frac{\ln 5^3 + \ln 7^2}{\ln(7/5)} = \frac{\ln 6125}{\ln(7/5)}.$$

TIP Give an exact answer, like $x = \dfrac{\ln 6125}{\ln(7/5)}$, whenever a question says *exact* or on a non-calculator paper (Paper 1). When a decimal is asked for, or on a calculator paper (Paper 2), round to three significant figures unless told otherwise. Three significant figures is the IB default, and rounding too early or giving too few figures costs accuracy marks.

1.5.3 When the unknown appears twice

Taking a logarithm gets nowhere on $2^{2x} - 10 \cdot 2^x + 16 = 0$, because x sits in two exponents and a sum of powers has no log law to break it up. This section covers two common patterns. When one power is the square of another, a **substitution** such as $u = a^x$ turns it into a quadratic. Solve the quadratic for u, then convert each root back. Reject any root with $u \leq 0$, since $a^x > 0$ for every real x (with $a > 0$) and no exponent can produce it. When the exponents differ by a constant, a common factor collapses the terms into one.

Worked through. To solve $2^{2x} - 10 \cdot 2^x + 16 = 0$, notice that the first term is the square of the second, since $2^{2x} = (2^x)^2$. Put $u = 2^x$:

$$u^2 - 10u + 16 = 0 \implies (u-2)(u-8) = 0,$$

so $u = 2$ or $u = 8$. Now convert each value back:

$$2^x = 2 \implies x = 1, \qquad 2^x = 8 \implies x = 3.$$

Both roots are positive, so both survive. A negative or zero value of u would have to be rejected instead of converted, since $2^x > 0$ for every real x and no exponent produces it.

EXAMPLE 1.11

Solve $3^{x+2} - 3^x = 72$.

Split the first exponent so that both terms carry 3^x: by the product law, $3^{x+2} = 3^x \cdot 3^2 = 9 \cdot 3^x$. The equation becomes

$$9 \cdot 3^x - 3^x = 72 \implies 8 \cdot 3^x = 72 \implies 3^x = 9,$$

so $x = 2$. Check: $3^4 - 3^2 = 81 - 9 = 72$. A constant added in the exponent is a constant factor outside it, which is what lets the two terms be gathered into one.

1.5.4 From words to an equation

Many growth and decay problems use one model,

$$A(t) = A_0 \times r^t,$$

where A_0 is the starting amount and r is the multiplier for each period. The wording gives you r. A rise of 6% per year means $r = 1.06$. A fall of 6% means $r = 0.94$. A quantity that doubles each period has $r = 2$.

When the change is continuous rather than in steps, the model uses base e:

$$A(t) = A_0\, e^{kt},$$

where k is the **continuous growth rate** met in §1.1: positive when the quantity grows, negative when it decays.

These are not two models but one, written twice. Since $r = e^{\ln r}$,

$$A_0\, r^t = A_0 \left(e^{\ln r}\right)^t = A_0\, e^{(\ln r)\, t},$$

so the two forms agree exactly when

$$k = \ln r.$$

A quantity that doubles each period has $k = \ln 2 \approx 0.693$; one falling 6% a year has $k = \ln 0.94 \approx -0.0619$, the negative sign marking decay. Use whichever form the question hands you: a percentage per period gives r, a rate described as *continuous* gives k.

Once the model is written, the logarithm provides the final step.

Decay is often described not by a rate but by a **half-life**: the time in which half of the quantity disappears. One half-life multiplies the amount by $\frac{1}{2}$, and a time t contains t/T half-lives, where T is the half-life itself. So

$$A(t) = A_0 \left(\tfrac{1}{2}\right)^{t/T}.$$

This is the same $A_0\, r^t$ model with $r = \frac{1}{2}$, and with time counted in half-lives instead of single periods. A **doubling time** works identically, with 2 in place of $\frac{1}{2}$.

EXAMPLE 1.12

Algae in a Finnish lake grow continuously at 12% per year, starting from 200 grams. When does the mass reach 500 grams?

Continuous growth uses base e, so $A(t) = 200\, e^{0.12t}$. Set it equal to the target:

$$200\, e^{0.12t} = 500 \implies e^{0.12t} = 2.5 \implies t = \frac{\ln 2.5}{0.12} \approx 7.64 \text{ years.}$$

The 12% here is a continuous rate, so it sits in the exponent as 0.12 directly, with no conversion to a yearly factor.

EXAMPLE 1.13

A medical isotope has a half-life of 6 hours: every 6 hours, half of it decays. An 80 mg sample is prepared. How long until only 12 mg remains?

Halving every 6 hours means $A(t) = 80\left(\tfrac{1}{2}\right)^{t/6}$, with t in hours. Set this equal to the target and take a logarithm:

$$80\left(\tfrac{1}{2}\right)^{t/6} = 12 \implies \left(\tfrac{1}{2}\right)^{t/6} = 0.15 \implies t = 6 \cdot \frac{\ln 0.15}{\ln 0.5} \approx 16.4 \text{ hours.}$$

The answer need not be a whole number of half-lives: 16.4 hours is a little over 2.7 of them, and the model runs continuously between them.

YOUR TURN 1.5 Solving Exponential Equations

Same base

38. Solve each equation, giving your answer as an exact value.

(a) $5^{2x} = 125$ (b) $4^x = 8^{x-1}$

(c) $9^{x+1} = 27^x$ (d) $2^x \cdot 4^{x+1} = 8$

Taking logarithms

39. GDC Solve each equation, giving your answer to 3 significant figures.

(a) $2^x = 10$ (b) $3^{x+1} = 20$

(c) $5^{2x} = 40$ (d) $7 \cdot 2^x = 200$

40. Solve each equation exactly. Give your answers to (a) and (b) in the form $\dfrac{\ln p}{\ln q}$.

(a) $3^{x+1} = 5^x$

(b) $2^{2x-1} = 7^{x+2}$

(c) $3 \cdot 2^x = 2 \cdot 3^x$

The unknown in two exponents

41. Solve each equation. Reject any value that no power can take. *(Hint: in (a), put $u = e^x$)*

(a) $e^{2x} - 5e^x + 6 = 0$

(b) $2^{2x} - 2 \cdot 2^x - 8 = 0$

(c) $3^{2x+1} - 8 \cdot 3^x - 3 = 0$

42. Solve each equation by taking out a common factor.

(a) $2^{x+3} - 2^x = 56$ (b) $5^{x+1} + 5^{x-1} = 130$

Growth and decay models

43. GDC The population of a town is $18\,000$ and rises by 4% each year.

(a) Write down a model for the population P after t years.

(b) Find the number of complete years after which the population first exceeds $30\,000$.

44. GDC A radioactive isotope has a half-life of 5.3 years. A sample has a mass of 200 g.

(a) Find the mass that remains after 12 years.

(b) Find how long it takes until only 15 g remains.

45. GDC A bacterial culture grows continuously. After t hours the number of bacteria is $N = 500\,e^{0.35t}$.

(a) Find the time taken for the number to reach 5000.

(b) Write the model in the form $N = 500\,r^{\,t}$, and hence find the percentage by which the culture grows each hour.

46. GDC Sea ice in a Finnish bay covers 1200 km^2 and shrinks by 6% each year. Find the number of years after which it has shrunk to 800 km^2.

Concept check

47. A student solves $3 \cdot 2^x = 48$ by writing $6^x = 48$, so $x = \dfrac{\ln 48}{\ln 6} \approx 2.16$. Identify the error, and solve the equation correctly.

48. A student solves $e^{2x} + e^x = 12$ by taking $\ln$ of each term, to get $2x + x = \ln 12$. Explain why this step is not valid, and solve the equation correctly.

Chapter 1 · Exponents & Logarithms

Reflections

TOK John Napier spent twenty years computing logarithms by hand and published his tables in 1614, chiefly to shorten the long multiplications of astronomy and navigation. He coined the word "logarithm", but his logarithms were not today's: they had no base in the modern sense and decreased as numbers increased. Henry Briggs, after visiting Napier, produced the base-10 tables that became standard (1617 and 1624). Does mathematics develop because we need it, or do we find uses for what mathematicians have already built?

TOK The exponent laws force $a^0 = 1$ for every $a \neq 0$, but 0^0 is often left undefined, because approaching it as a limit along different routes gives different answers, while algebra and combinatorics (the mathematics of counting) usually set $0^0 = 1$. Does leaving 0^0 undefined reveal a limitation of mathematics or the precision of mathematics?

IM Base-10 logarithms feel natural to us because we count in base 10. But the choice is cultural, not mathematical. The Babylonians used base 60 (traces of which survive in our 60-minute hour and 360-degree circle). The notation is cultural too. The word *algebra* comes from *al-jabr*, a technique described by al-Khwarizmi in ninth-century Baghdad, whose algebra was written entirely in words; the raised exponent used in this chapter was popularised by René Descartes only in 1637. What does it mean for a mathematical tool to be "natural"?

RW The decibel (dB) scale for sound is logarithmic: the sound level is $10 \log_{10}(I/I_0)$, comparing a sound's intensity I to a fixed reference intensity I_0, so each increase of 10 dB means ten times the intensity. The stellar magnitude scale in astronomy and the octave system in music work the same way. In each case, human perception responds approximately to the logarithm of the physical quantity, not to the quantity itself.

End-of-Chapter Problems
Chapter 1: Exponents & Logarithms

1. Solve $3^{2x+1} = 27^{x-1}$. [3]

2.

(a) Write $\sqrt{50} + \sqrt{18} - \sqrt{8}$ in the form $k\sqrt{2}$, where k is an integer. [2]

(b) Express $\dfrac{4+\sqrt{3}}{2-\sqrt{3}}$ in the form $a + b\sqrt{3}$, where $a, b \in \mathbb{Z}$. [3]

(c) Given that $27^x = 3\sqrt{3}$, find the exact value of x. [2]

3. Show that the equation $3^{x+2} + 3^x = k$ has a real solution only when $k > 0$, and find this solution in terms of k. [4]

4.

(a) Simplify $\left(2x^{-2}y^3\right)^3 \times \left(4x^4y^{-1}\right)^{-1}$, giving your answer with positive indices. [3]

(b) Solve $8^x = 2^{x+6}$. [2]

(c) Show that $\left(a^{1/2} + a^{-1/2}\right)^2 = a + 2 + a^{-1}$, for $a > 0$. [2]

5. GDC The mass of the Earth is 5.97×10^{24} kg and the mass of the Moon is 7.35×10^{22} kg.

(a) Find how many times heavier the Earth is than the Moon, to 3 significant figures. [2]

(b) Find their combined mass, giving your answer in standard form to 3 significant figures. [2]

6. Without a calculator, show that $2 < \log_2 5 < 3$. Hence find the integer part of $\log_2 40$. [4]

7. Given that $\log_a x = 3$ and $\log_a y = 5$, find each of the following.

(a) $\log_a(x^2y)$ [1]

(b) $\log_a \sqrt{xy}$ [2]

(c) $\log_a \left(\dfrac{x}{y^2}\right)$ [1]

8. Let $\log_a 2 = p$ and $\log_a 3 = q$.

(a) Write $\log_a 6$ in terms of p and q. [1]

(b) Write $\log_a \sqrt{12}$ in terms of p and q. [2]

(c) Write $\log_a \dfrac{4}{9}$ in terms of p and q. [2]

9.

(a) Show that $\log_a \sqrt[n]{x} = \dfrac{1}{n}\log_a x$. [2]

(b) Hence solve $\log_5 \sqrt[3]{x} + \log_{125} x = 2$. [4]

10. Prove that $\log_a b \cdot \log_b c \cdot \log_c a = 1$ for all valid bases and arguments. Hence evaluate $\log_2 3 \cdot \log_3 5 \cdot \log_5 8$ without a calculator. [4]

11. Given that $\log_2 3 = a$, express each of the following in terms of a:

(a) $\log_2 9$ [1]

(b) $\log_4 3$ [2]

(c) $\log_8 27$ [2]

(d) $\log_3 8$ [2]

12. Solve $\log_3(x+5) - \log_3(x-1) = 2$. [4]

13. Solve $\log_6(x+2) + \log_6(x-3) = 1$, justifying any rejection of solutions. [4]

14. Solve the simultaneous equations:

$$\log_2 x + \log_2 y = 5, \qquad \log_2(x-y) = 3.$$

[5]

15. Solve $\log_x 8 = \log_4 x$. [5]

16. GDC Solve $\log_2(x^2 - 2x) = \log_4(x+4)$. [5]

17. GDC

(a) Express $\log_3 x - 2 = \log_9(x+6)$ as a quadratic equation in x, using change of base. [3]

(b) Hence find the value of x. [2]

18. Solve $\log_5(x^2 - 2x + 1) = 2\log_5(x-1) + 1$, or show that no solution exists. [3]

19. Solve $3^{2x-1} = 4^{x+2}$, giving your answer in exact logarithmic form. [4]

20. Find the exact solution to $e^{2x} = 3e^x + 10$. *(Hint: let $u = e^x$)* [4]

21. Solve the equation $9^x - 4 \cdot 3^{x+1} + 27 = 0$. *(Hint: let $u = 3^x$; note $9^x = u^2$)* [5]

22. The reciprocal of a logarithm is a logarithm with its base and argument swapped.

(a) Let $N = a^k$, where $a > 0$, $a \neq 1$, $N > 0$ and $N \neq 1$. Write $\log_a N$ and $\log_N a$ in terms of k, and hence show that $\dfrac{1}{\log_a N} = \log_N a$. Explain why $N = 1$ must be excluded. [3]

(b) Hence evaluate $\dfrac{1}{\log_2 36} + \dfrac{1}{\log_3 36}$ without a calculator. [4]

23. Solve $\log_2 x + \log_x 2 = \dfrac{5}{2}$. *(Hint: let $t = \log_2 x$ and use question 22)* [6]

24. Consider the equation $(\log_3 x)^2 = \log_3 x^2 + 3$.

(a) Explain why the right-hand side equals $2\log_3 x + 3$, and state the restriction on x this requires. [2]

(b) Hence solve the equation. [4]

(c) The expression $\log_3 x^2$ is defined for every non-zero x, yet negative values of x cannot be solutions. Explain why. [2]

25. Consider the equation $\log_2(\log_3(\log_2 x)) = 0$.

(a) Find x. [4]

(b) Working outwards from the innermost logarithm, state the condition each layer imposes on x, and hence give the values of x for which the left-hand side is defined at all. [4]

26. Evaluate

$$\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \ \cdots \ \cdot \log_{31} 32,$$

the product of all thirty factors, without a calculator. Justify your method. [5]

27. Let a, b and c be positive, with a, b, c and ab all different from 1.

(a) Show that $\log_{ab} c = \dfrac{\log_a c \cdot \log_b c}{\log_a c + \log_b c}$. [5]

(b) Verify your identity with $a = 2$, $b = 5$ and $c = 7$, giving both sides to four decimal places. [2]

28. Solve the simultaneous equations

$$\log_x y + \log_y x = \frac{10}{3}, \qquad xy = 81,$$

where x and y are positive and neither is 1. [7]

29. GDC The number of bacteria in a culture doubles every 3 hours. Initially there are 500 bacteria.

(a) Write an expression for the number of bacteria after t hours. [2]

(b) Find the time, to the nearest minute, when there are 50 000 bacteria. [3]

30. GDC The population of a city is modelled by $P(t) = 250\,000 \times e^{0.032t}$, where t is the number of years after the start of 2020.

(a) Write down the population in 2020. [1]
(b) Find the population in 2030, to the nearest thousand. [2]
(c) Find the year in which the population first exceeds 400 000. [3]

31. GDC The value V dollars of a delivery van t years after it is bought is modelled by $V = 45\,000 \times (0.82)^t$.

(a) Write down the value of the van when it is bought. [1]
(b) Find the value after 5 years, to the nearest dollar. [2]
(c) Find the least number of complete years after which the value is below $10 000. [3]

32. GDC The pH of a solution is defined by pH $= -\log_{10}[\text{H}^+]$, where $[\text{H}^+]$ is the hydrogen ion concentration in mol/L.

(a) A solution has $[\text{H}^+] = 3.5 \times 10^{-4}$ mol/L. Find its pH to 2 decimal places. [2]
(b) A solution has pH $= 8.3$. Find its hydrogen ion concentration in standard form. [2]
(c) Solution A has pH $= 3$ and solution B has pH $= 7$. Find how many times more acidic solution A is than solution B. [2]

33. GDC A radioactive substance decays so that its mass M grams after t years satisfies $M = M_0 e^{-kt}$, where M_0 and k are positive constants. After 10 years the mass is halved.

(a) Show that $k = \dfrac{\ln 2}{10}$. [2]
(b) Find the percentage of the original mass remaining after 25 years. [3]
(c) Find the time for the mass to reduce to 10% of its original value. [3]

34. GDC This question is about writing very large and very small quantities in standard form.

(a) A human cell has diameter about 2×10^{-5} m, and a water molecule about 2.8×10^{-10} m. Find how many times wider the cell is, giving your answer in standard form to three significant figures. [2]
(b) The observable universe is about 8.8×10^{26} m across, and the Planck length, the scale at which current physical theories are expected to break down, is about 1.6×10^{-35} m. Find the ratio of the two, in standard form. [2]
(c) A sheet of paper is 0.1 mm thick. It is folded in half, then in half again, and so on. Find the thickness after 20 folds, in metres and in standard form. [2]
(d) Find the least number of folds needed for the thickness to exceed the distance to the Moon, 3.84×10^8 m. [4]
(e) Your answer to part (d) may be smaller than you expected. Explain why, referring to how the thickness grows. [2]

35.

(a) Prove the change of base rule $\log_a b = \dfrac{\log_c b}{\log_c a}$, for valid bases a and c. *(Hint: let $x = \log_a b$, so $a^x = b$, then take $\log_c$ of both sides)* [3]

(b) Hence show that $\log_{27} x = \frac{1}{3}\log_3 x$. [2]

(c) Solve $\log_3 x + \log_9 x + \log_{27} x = 11$. [4]

(d) Solve $\log_x 2 + \log_{x^2} 2 = 3$, giving an exact answer. [5]

Challenge Questions

36. GDC **Decibels.** The loudness L of a sound in decibels is $L = 10\log_{10}\left(\dfrac{I}{I_0}\right)$, where I is the intensity in W/m^2 and $I_0 = 10^{-12}$ W/m^2 is the threshold of hearing. The scale is logarithmic because human hearing spans a range of intensities far too wide to put on a linear axis.

(a) A concert has loudness 110 dB. Find the intensity I. [2]

(b) A whisper has loudness 30 dB. Show that the concert is 10^8 times as intense as the whisper. [2]

(c) Show that doubling the intensity of *any* sound raises its loudness by $10\log_{10} 2$ dB, whatever the original level was. Give this value to two decimal places. [4]

(d) Hence find how many identical sources, playing together, are needed to raise the level by 10 dB. [3]

(e) Intensity obeys an inverse-square law: at distance r from a source, $I \propto \dfrac{1}{r^2}$. Show that doubling your distance reduces the level by a fixed number of decibels, independent of where you started, and find it. [4]

(f) A workplace safety rule allows 8 hours of exposure at 85 dB, and halves the permitted time for every 3 dB above that. Find the permitted time at 100 dB, and explain what part (c) has to do with the choice of 3 dB. [4]

(g) Two sources, of levels L_1 and L_2 dB, play together, and their intensities add. Show that the combined level is

$$L = 10\log_{10}\left(10^{L_1/10} + 10^{L_2/10}\right),$$

and find the combined level of a 90 dB source and an 80 dB source. [3]

(h) Let $L_1 \geq L_2$ and $D = L_1 - L_2$. Show that $L - L_1 = 10\log_{10}\left(1 + 10^{-D/10}\right)$, tabulate this rise for $D = 0, 3, 6, 10, 20$, and decide, with a reason, how much quieter a second source must be before it can be ignored. Suggest how this could be extended into an exploration. [3]

37. GDC **Counting digits, and the first digit.** This investigation uses logarithms to count digits, and then to explain a pattern that auditors use to detect faked data. Throughout, $\lfloor t \rfloor$ is the greatest integer not exceeding t, and the fractional part of t is $t - \lfloor t \rfloor$.

(a) Let N be a positive integer. Show that N has exactly $\lfloor \log_{10} N \rfloor + 1$ digits. [3]

(b) Hence find the number of digits in 2^{1000}. [2]

(c) Let d be one of the digits $1, 2, \ldots, 9$. Prove that the first digit of N is d exactly when the fractional part of $\log_{10} N$ lies in the interval $[\log_{10} d,\ \log_{10}(d+1))$. Verify this for $N = 2^7$. [4]

(d) Prove that $\log_{10} 2$ is irrational. Hence show that no two of the numbers $\log_{10}(2^n)$, for $n = 1, 2, 3, \ldots$, have the same fractional part. [4]

(e) Assume the fractional parts of $\log_{10}(2^n)$ are spread evenly over $[0, 1)$ as n increases. (Part (d) shows that they never repeat; that they spread evenly is a further theorem, which you may use.) Show that the proportion of powers of 2 whose first digit is d is $\log_{10}\left(1 + \dfrac{1}{d}\right)$. [4]

(f) Evaluate this proportion for $d = 1$ and for $d = 9$. Then count the first digits of 2^1, 2^2, …, 2^{30} on your GDC and compare. [4]

(g) The pattern in part (e) is known as Benford's law, and it is also reported for data such as the populations of towns. Comment on how well your counts in part (f) support it, and why a sample of 30 might mislead. Then name one kind of real data you would expect to follow the law and one you would not, giving a reason for each, as the starting point of an exploration. [3]

38. GDC **Trapping e.** For each integer $n \geq 1$ let

$$a_n = \left(1 + \frac{1}{n}\right)^n \qquad \text{and} \qquad b_n = \left(1 + \frac{1}{n}\right)^{n+1}.$$

You may use *Bernoulli's inequality*: for every integer $m \geq 2$ and every real $x \geq -1$ with $x \neq 0$, $(1+x)^m > 1 + mx$. (It can be proved by induction, the method of Ch. 3.) You may also use the fact that an increasing sequence that is bounded above has a limit, and so does a decreasing sequence that is bounded below.

(a) Find a_n and b_n for $n = 1, 2, 5, 10, 100$, giving values to four decimal places where they are not exact. Make a conjecture about how each sequence behaves. [3]

(b) Show that $\dfrac{a_{n+1}}{a_n} = \left(1 + \dfrac{1}{n}\right)\left(1 - \dfrac{1}{(n+1)^2}\right)^{n+1}$, and use Bernoulli's inequality to prove that $a_{n+1} > a_n$ for all $n \geq 1$. [5]

(c) Show that, for $n \geq 2$, $\dfrac{b_{n-1}}{b_n} = \dfrac{n}{n+1}\left(1 + \dfrac{1}{n^2 - 1}\right)^n$, and prove that $b_n < b_{n-1}$. [5]

(d) Show that $b_n - a_n = \dfrac{a_n}{n}$. Explain why $\{a_n\}$ and $\{b_n\}$ both have limits, and why the two limits are equal. This common limit is e; deduce that $a_n < e < b_n$ for every $n \geq 1$. [4]

(e) Use $n = 1000$ to write down an interval that contains e, and state its width. [2]

(f) Compute $n(e - a_n)$ for $n = 10$, 100 and 1000, and conjecture its limit. Explain what your conjecture says about how many correct decimal places of e are gained each time n is multiplied by 10, and suggest a way to obtain e more quickly that could be investigated further. [3]